How to study Chapter 1

What you should learn

In this chapter you will learn the following skills and concepts:

- How to sketch the graph of an equation
- How to solve linear equations, quadratic equations, polynomial equations, radical equations, and absolute value equations
- How to perform operations with complex numbers
- How to solve linear inequalities, polynomial inequalities, rational inequalities, and inequalities involving absolute value

Important Vocabulary

As you encounter each new vocabulary term in this chapter, add the term and its definition to your notebook glossary.

Equation in two variables (p. 78)
Solution of equation in two variables (p. 78)
Graph of an equation (p. 78)
Intercepts (p. 80)
Symmetry (p. 80)
Circle (p. 83)
Equation in one variable (p. 88)
Solution of equation in one variable (p. 88)
Identity equation (p. 88)
Conditional equation (p. 88)
Linear equation in one variable (p. 88)
Equivalent equations (p. 89)
Extraneous solution (p. 91)
Quadratic equation (p. 109)
Quadratic Formula (p. 112)
Discriminant (p. 113)
Position equation (p. 115)
Complex number (p. 123)
Imaginary number (p. 123)
Pure imaginary number (p. 123)
Complex conjugates (p. 126)
Principal square root of a negative number (p. 127)
Polynomial equation (p. 130)
Solution of an inequality (p. 141)
Graph of an inequality (p. 141)
Linear inequality in one variable (p. 143)
Double inequality (p. 144)
Critical numbers (p. 151)
Test intervals (p. 151)

Study Tools

Learning Objectives in each section
Chapter Summary (p. 161)
Review Exercises (pp. 162–165)
Chapter Test (p. 166)

Additional Resources

Study and Solutions Guide
Interactive College Algebra
Videotapes/DVD for Chapter 1
College Algebra Website
Student Success Organizer

1

Equations and Inequalities

1.1 Graphs of Equations
1.2 Linear Equations in One Variable
1.3 Modeling with Linear Equations
1.4 Quadratic Equations
1.5 Complex Numbers
1.6 Other Types of Equations
1.7 Linear Inequalities in One Variable
1.8 Other Types of Inequalities

1.1 Graphs of Equations

▶ **What you should learn**

- How to sketch graphs of equations
- How to find x- and y-intercepts of graphs of equations
- How to use symmetry to sketch graphs of equations
- How to find equations and sketch graphs of circles
- How to use graphs of equations in solving real-life problems

▶ **Why you should learn it**

The graph of an equation can help you see relationships between real-life quantities. For example, in Exercise 67 on page 87, a graph can be used to estimate the life expectancies of children who are born in the years 2005 and 2010.

Bruce Forster/Tony Stone Images

The Graph of an Equation

In Section P.7, you used a coordinate system to represent graphically the relationship between two quantities. There, the graphical picture consisted of a collection of points in a coordinate plane.

Frequently, a relationship between two quantities is expressed as an **equation in two variables.** For instance, $y = 7 - 3x$ is an equation in x and y. An ordered pair (a, b) is a **solution** or **solution point** of an equation in x and y if the equation is true when a is substituted for x and b is substituted for y. For instance, $(1, 4)$ is a solution of $y = 7 - 3x$ because $4 = 7 - 3(1)$ is a true statement.

In this section you will review some basic procedures for sketching the graph of an equation in two variables. The **graph of an equation** is the set of all points that are solutions of the equation.

Example 1 ▶ Sketching the Graph of an Equation

Sketch the graph of $y = 7 - 3x$.

Solution

The simplest way to sketch the graph of an equation is the *point-plotting method.* With this method, you construct a table of values that consists of several solution points of the equation. For instance, when $x = 0$,

$$y = 7 - 3(0)$$
$$= 7$$

which implies that $(0, 7)$ is a solution point of the graph.

x	$y = 7 - 3x$	(x, y)
0	7	$(0, 7)$
1	4	$(1, 4)$
2	1	$(2, 1)$
3	-2	$(3, -2)$
4	-5	$(4, -5)$

From the table, it follows that

$$(0, 7), (1, 4), (2, 1), (3, -2), \text{ and } (4, -5)$$

are solution points of the equation. After plotting these points, you can see that they appear to lie on a line, as shown in Figure 1.1. The graph of the equation is the line that passes through the five plotted points.

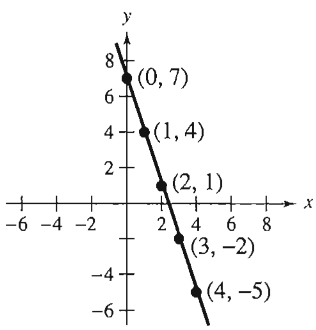


FIGURE 1.1

The icon identifies examples and concepts related to features of the Learning Tools CD-ROM and the *Interactive* and *Internet* versions of this text. For more details see the chart on pages *xix–xxiii*.

The *Interactive* CD-ROM and *Internet* versions of this text offer a Try It for each example in the text.

Example 2 ▶ Sketching the Graph of an Equation

Sketch the graph of

$$y = x^2 - 2.$$

Solution

Begin by constructing a table of values.

x	-2	-1	0	1	2	3
$y = x^2 - 2$	2	-1	-2	-1	2	7
(x, y)	$(-2, 2)$	$(-1, -1)$	$(0, -2)$	$(1, -1)$	$(2, 2)$	$(3, 7)$

Next, plot the points given in the table, as shown in Figure 1.2. Finally, connect the points with a smooth curve, as shown in Figure 1.3.

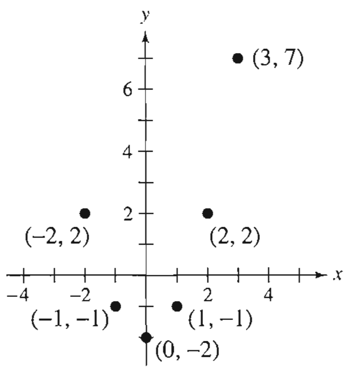


FIGURE 1.2

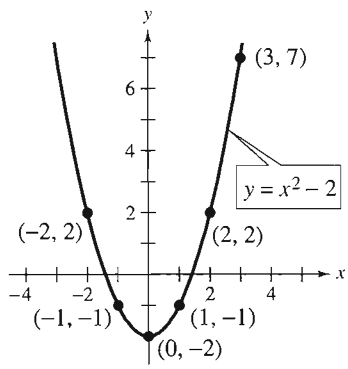


FIGURE 1.3

STUDY TIP

One of your goals in this course is to learn to classify the basic shape of a graph from its equation. For instance, you will learn that the *linear equation* in Example 1 has the form

$$y = mx + b$$

and its graph is a line. Similarly, the *quadratic equation* in Example 2 has the form

$$y = ax^2 + bx + c$$

and its graph is a parabola.

The point-plotting technique demonstrated in Examples 1 and 2 is easy to use, but it has some shortcomings. With too few solution points, you can misrepresent the graph of an equation. For instance, if only the four points

$$(-2, 2), (-1, -1), (1, -1), \text{ and } (2, 2)$$

in Figure 1.2 were plotted, any one of the three graphs in Figure 1.4 would be reasonable.

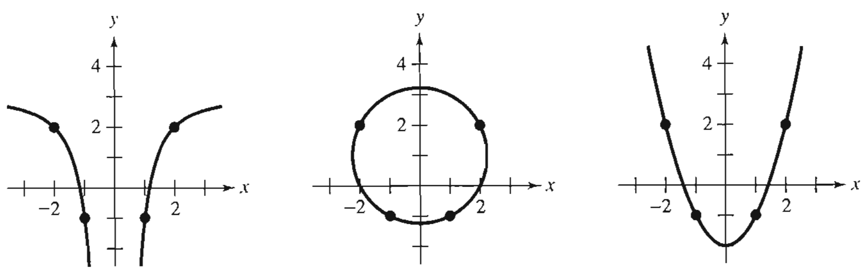
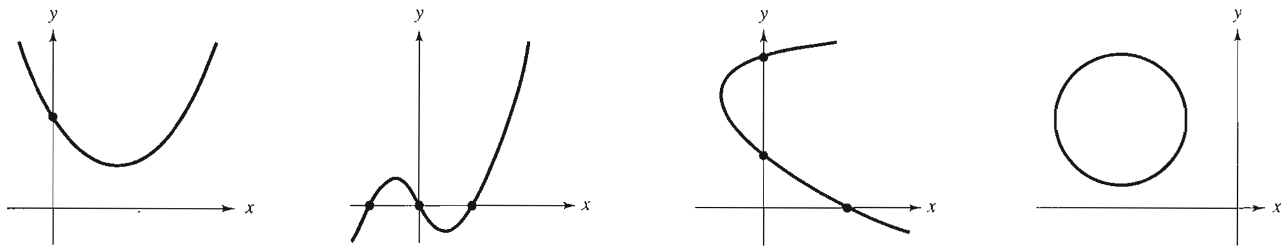


FIGURE 1.4

Intercepts of a Graph

It is often easy to determine the solution points that have zero as either the x-coordinate or the y-coordinate. These points are called **intercepts** because they are the points at which the graph intersects the x- or y-axis. It is possible for a graph to have no intercepts, one intercept, or several intercepts, as shown in Figure 1.5.



No x-intercept
One y-intercept

Three x-intercepts
One y-intercept

One x-intercept
Two y-intercepts

No intercepts

FIGURE 1.5

Note that an x-intercept is written as the ordered pair $(x, 0)$ and a y-intercept is written as the ordered pair $(0, y)$.

Example 3 ▶ Identifying x- and y-Intercepts

Identify the x- and y-intercepts of the graph of $y = x^3 + 1$ shown in Figure 1.6.

The *Interactive* CD-ROM and *Internet* versions of this text offer a Quiz for every section of the text.

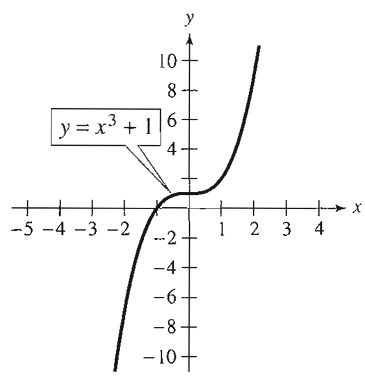


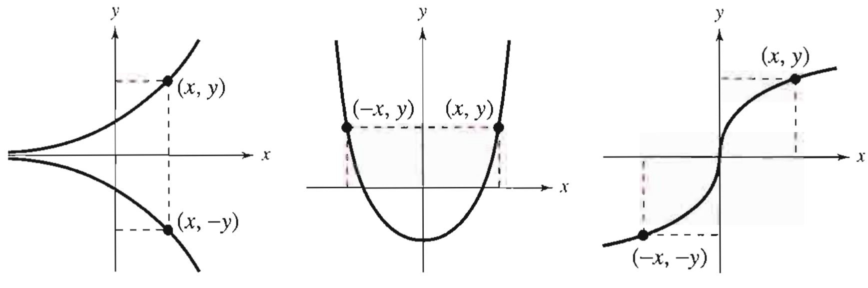
FIGURE 1.6

Solution

From the graph, you can see that the graph of the equation $y = x^3 + 1$ has an x-intercept (where y is zero) at $(-1, 0)$ and a y-intercept (where x is zero) at $(0, 1)$.

Symmetry

Graphs of equations can have **symmetry** with respect to one of the coordinate axes or with respect to the origin. Symmetry with respect to the x-axis means that if the Cartesian plane were folded along the x-axis, the portion of the graph above the x-axis would coincide with the portion below the x-axis. Symmetry with respect to the y-axis or the origin can be described in a similar manner, as shown in Figure 1.7.



x-Axis symmetry *y-Axis symmetry* *Origin symmetry*

FIGURE 1.7

Knowing the symmetry of a graph *before* attempting to sketch it is helpful, because then you need only half as many solution points to sketch the graph. There are three basic types of symmetry, described as follows.

Graphical Tests for Symmetry

1. A graph is **symmetric with respect to the x-axis** if, whenever (x, y) is on the graph, $(x, -y)$ is also on the graph.
2. A graph is **symmetric with respect to the y-axis** if, whenever (x, y) is on the graph, $(-x, y)$ is also on the graph.
3. A graph is **symmetric with respect to the origin** if, whenever (x, y) is on the graph, $(-x, -y)$ is also on the graph.

Example 4 ▶ Testing for Symmetry

The graph of $y = x^2 - 2$ is symmetric with respect to the y-axis because the point $(-x, y)$ is also on the graph of $y = x^2 - 2$. (See Figure 1.8.) The table at the left confirms that the graph is symmetric with respect to the y-axis.

x	y	(x, y)
−3	7	(−3, 7)
−2	2	(−2, 2)
−1	−1	(−1, −1)
1	−1	(1, −1)
2	2	(2, 2)
3	7	(3, 7)

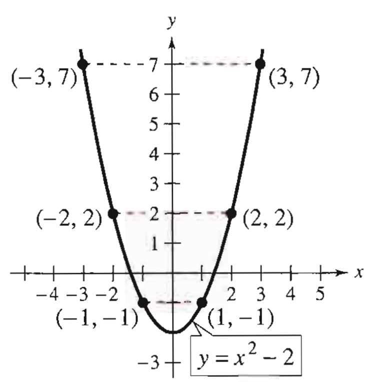


FIGURE 1.8 *y-Axis symmetry*

Algebraic Tests for Symmetry

1. The graph of an equation is symmetric with respect to the x-axis if replacing y with $-y$ yields an equivalent equation.
2. The graph of an equation is symmetric with respect to the y-axis if replacing x with $-x$ yields an equivalent equation.
3. The graph of an equation is symmetric with respect to the origin if replacing x with $-x$ and y with $-y$ yields an equivalent equation.

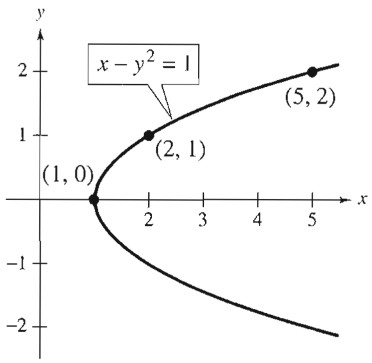


FIGURE 1.9

Example 5 ▶ Using Symmetry as a Sketching Aid

Use symmetry to sketch the graph of

$$x - y^2 = 1.$$

Solution

Of the three tests for symmetry, the only one that is satisfied is the test for x-axis symmetry because $x - (-y)^2 = 1$ is equivalent to $x - y^2 = 1$. So, the graph is symmetric with respect to the x-axis. Using symmetry, you need only to find the solution points above the x-axis and then reflect them to obtain the graph, as shown in Figure 1.9.

y	$x = y^2 + 1$	(x, y)
0	1	(1, 0)
1	2	(2, 1)
2	5	(5, 2)

STUDY TIP

Notice that when creating the table in Example 5, it is easier to choose y-values and then find the corresponding x-values of the ordered pairs.

Example 6 ▶ Sketching the Graph of an Equation

Sketch the graph of

$$y = |x - 1|.$$

Solution

This equation fails all three tests for symmetry and consequently its graph is not symmetric with respect to either axis or to the origin. The absolute value sign indicates that y is always nonnegative. Create a table of values and plot the points as shown in Figure 1.10. From the table, you can see that $x = 0$ when $y = 1$. So, the y-intercept is $(0, 1)$. Similarly, $y = 0$ when $x = 1$. So, the x-intercept is $(1, 0)$.

x	-2	-1	0	1	2	3	4
$y = \|x - 1\|$	3	2	1	0	1	2	3
(x, y)	$(-2, 3)$	$(-1, 2)$	$(0, 1)$	$(1, 0)$	$(2, 1)$	$(3, 2)$	$(4, 3)$

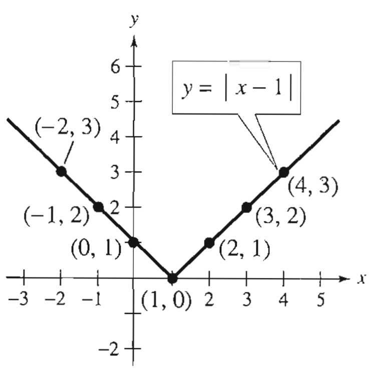


FIGURE 1.10

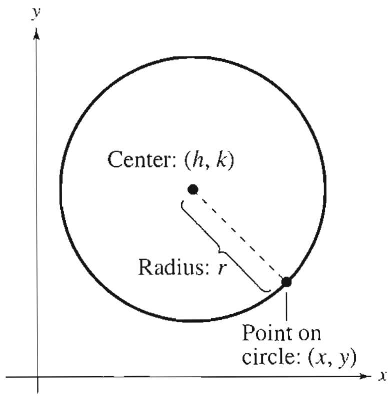


FIGURE 1.11

Throughout this course, you will learn to recognize several types of graphs from their equations. For instance, you will learn to recognize that the graph of a second-degree equation of the form

$$y = ax^2 + bx + c$$

is a parabola (see Example 2). Another easily recognized graph is that of a **circle.**

Circles

Consider the circle shown in Figure 1.11. A point (x, y) is on the circle if and only if its distance from the center (h, k) is r. By the Distance Formula,

$$\sqrt{(x-h)^2 + (y-k)^2} = r.$$

By squaring each side of this equation, you obtain the **standard form of the equation of a circle.**

Standard Form of the Equation of a Circle

The point (x, y) lies on the circle of radius r and center (h, k) if and only if

$$(x-h)^2 + (y-k)^2 = r^2.$$

From this result, you can see that the standard form of the equation of a circle *with its center at the origin,* $(h, k) = (0, 0)$, is simply

$$x^2 + y^2 = r^2. \qquad \text{Circle with center at origin}$$

STUDY TIP

To find the correct h and k, it may be helpful to rewrite the quantities $(x+1)^2$ and $(y-2)^2$, using subtraction.

$$(x+1)^2 = [x-(-1)]^2,$$
$$h = -1$$
$$(y-2)^2 = [y-(2)]^2,$$
$$k = 2$$

Example 7 ▶ Finding the Equation of a Circle

The point $(3, 4)$ lies on a circle whose center is at $(-1, 2)$, as shown in Figure 1.12. Write the standard form of the equation of this circle.

Solution

The radius of the circle is the distance between $(-1, 2)$ and $(3, 4)$.

$$r = \sqrt{(x-h)^2 + (y-k)^2} \qquad \text{Distance Formula}$$
$$r = \sqrt{[3-(-1)]^2 + (4-2)^2} \qquad \text{Substitute for } x, y, h, \text{ and } k.$$
$$= \sqrt{4^2 + 2^2} \qquad \text{Simplify.}$$
$$= \sqrt{16+4} \qquad \text{Simplify.}$$
$$= \sqrt{20} \qquad \text{Radius}$$

Using $(h, k) = (-1, 2)$ and $r = \sqrt{20}$, the equation of the circle is

$$(x-h)^2 + (y-k)^2 = r^2 \qquad \text{Equation of circle}$$
$$[x-(-1)]^2 + (y-2)^2 = (\sqrt{20})^2 \qquad \text{Substitute for } h, k, \text{ and } r.$$
$$(x+1)^2 + (y-2)^2 = 20. \qquad \text{Standard form}$$

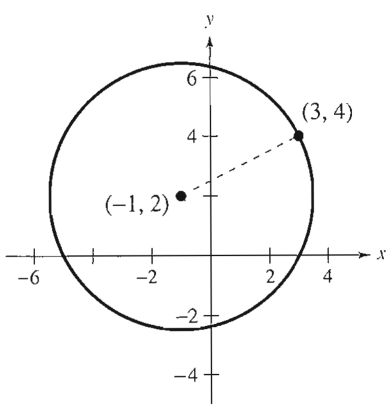


FIGURE 1.12

STUDY TIP

You should develop the habit of using at least two approaches to solve every problem. This helps build your intuition and helps you check that your answer is reasonable.

Application

In this course, you will learn that there are many ways to approach a problem. Three common approaches are illustrated in Example 8.

A *Numerical Approach:* Construct and use a table.
A *Graphical Approach:* Draw and use a graph.
An *Analytical Approach:* Use the rules of algebra.

Example 8 ▶ Recommended Weight

The median recommended weight y (in pounds) for men of medium frame who are 25 to 59 years old can be approximated by the mathematical model

$$y = 0.073x^2 - 6.99x + 289.0, \quad 62 \le x \le 76$$

where x is the man's height in inches. (Source: Metropolitan Life Insurance Company)

a. Construct a table of values that shows the median recommended weights for men with heights of 62, 64, 66, 68, 70, 72, 74, and 76 inches.

b. Use the table of values to sketch a graph of the model. Then use the graph to estimate *graphically* the median recommended weight for a man whose height is 71 inches.

c. Use the model to confirm *analytically* the estimate you found in part (b).

Solution

a. You can use a calculator to complete the table, as shown at the left.

Height, x	Weight, y
62	136.2
64	140.6
66	145.6
68	151.2
70	157.4
72	164.2
74	171.5
76	179.4

b. The table of values can be used to sketch the graph of the function, as shown in Figure 1.13. From the graph, you can estimate that a height of 71 inches corresponds to a weight of about 161 pounds.

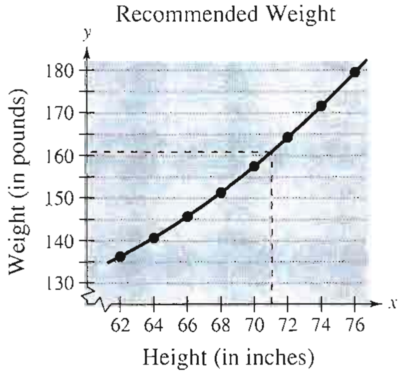


FIGURE 1.13

c. To confirm algebraically the estimate found in part (b), you can substitute 71 for x in the model.

$$y = 0.073(71)^2 - 6.99(71) + 289.0 \approx 160.70$$

So, the graphical estimate of 161 pounds is fairly good.

1.1 Exercises

The *Interactive* CD-ROM and *Internet* versions of this text contain step-by-step solutions to all odd-numbered exercises. They also provide Tutorial Exercises for additional help.

In Exercises 1–4, determine whether each point lies on the graph of the equation.

Equation	*Points*	
1. $y = \sqrt{x + 4}$	(a) $(0, 2)$	(b) $(5, 3)$
2. $y = x^2 - 3x + 2$	(a) $(2, 0)$	(b) $(-2, 8)$
3. $y = 4 - \lvert x - 2 \rvert$	(a) $(1, 5)$	(b) $(6, 0)$
4. $y = \frac{1}{3}x^3 - 2x^2$	(a) $\left(2, -\frac{16}{3}\right)$	(b) $(-3, 9)$

In Exercises 5–8, complete the table. Use the resulting solution points to sketch the graph of the equation.

5. $y = -2x + 5$

x	-1	0	1	2	$\frac{5}{2}$
y					
(x, y)					

6. $y = \frac{3}{4}x - 1$

x	-2	0	1	$\frac{4}{3}$	2
y					
(x, y)					

7. $y = x^2 - 3x$

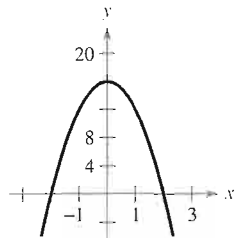
x	-1	0	1	2	3
y					
(x, y)					

8. $y = 5 - x^2$

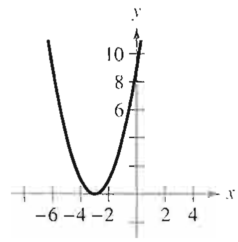
x	-2	-1	0	1	2
y					
(x, y)					

In Exercises 9–12, find the *x*- and *y*-intercepts of the graph of the equation.

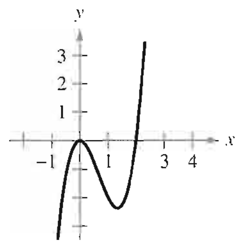
9. $y = 16 - 4x^2$



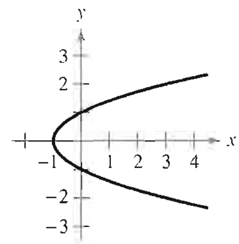
10. $y = (x + 3)^2$



11. $y = 2x^3 - 4x^2$



12. $y^2 = x + 1$



In Exercises 13–20, use the algebraic tests to check for symmetry with respect to both axes and the origin.

13. $x^2 - y = 0$

14. $x - y^2 = 0$

15. $y = x^3$

16. $y = x^4 - x^2 + 3$

17. $y = \dfrac{x}{x^2 + 1}$

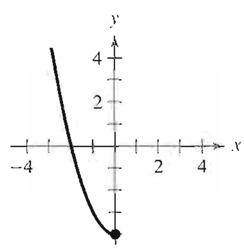
18. $y = \dfrac{1}{x^2 + 1}$

19. $xy^2 + 10 = 0$

20. $xy = 4$

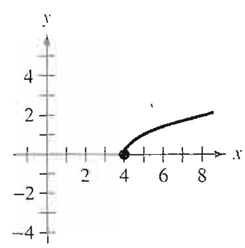
In Exercises 21–24, assume that the graph has the indicated type of symmetry. Sketch the complete graph of the equation. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

21.



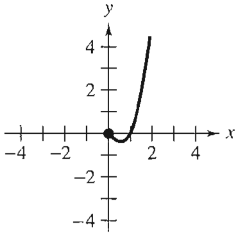
y-Axis symmetry

22.



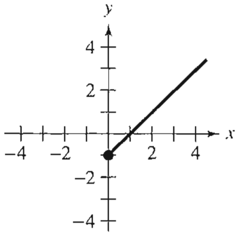
x-Axis symmetry

23.



Origin symmetry

24.



y-Axis symmetry

In Exercises 25–36, use symmetry to sketch the graph of the equation.

25. $y = -3x + 1$

26. $y = 2x - 3$

27. $y = x^2 - 2x$

28. $y = -x^2 - 2x$

29. $y = x^3 + 3$

30. $y = x^3 - 1$

31. $y = \sqrt{x - 3}$

32. $y = \sqrt{1 - x}$

33. $y = |x - 6|$

34. $y = 1 - |x|$

35. $x = y^2 - 1$

36. $x = y^2 - 5$

In Exercises 37–48, use a graphing utility to graph the equation. Use a standard setting. Approximate any intercepts.

37. $y = 3 - \frac{1}{2}x$

38. $y = \frac{2}{3}x - 1$

39. $y = x^2 - 4x + 3$

40. $y = x^2 + x - 2$

41. $y = \dfrac{2x}{x - 1}$

42. $y = \dfrac{4}{x^2 + 1}$

43. $y = \sqrt[3]{x}$

44. $y = \sqrt[3]{x + 1}$

45. $y = x\sqrt{x + 6}$

46. $y = (6 - x)\sqrt{x}$

47. $y = |x + 3|$

48. $y = 2 - |x|$

In Exercises 49–56, write the standard form of the equation of the specified circle.

49. Center: $(0, 0)$; radius: 4

50. Center: $(0, 0)$; radius: 5

51. Center: $(2, -1)$; radius: 4

52. Center: $(-7, -4)$; radius: 7

53. Center: $(-1, 2)$; solution point: $(0, 0)$

54. Center: $(3, -2)$; solution point: $(-1, 1)$

55. Endpoints of a diameter: $(0, 0)$, $(6, 8)$

56. Endpoints of a diameter: $(-4, -1)$, $(4, 1)$

In Exercises 57–62, find the center and radius of the circle, and sketch its graph.

57. $x^2 + y^2 = 25$

58. $x^2 + y^2 = 16$

59. $(x - 1)^2 + (y + 3)^2 = 9$

60. $x^2 + (y - 1)^2 = 1$

61. $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{9}{4}$

62. $(x - 2)^2 + (y + 1)^2 = 3$

63. ***Depreciation*** A manufacturing plant purchases a new molding machine for \$225,000. The depreciated value y after t years is

$$y = 225{,}000 - 20{,}000t, \qquad 0 \le t \le 8.$$

Sketch the graph of the equation.

64. ***Consumerism*** You purchase a jet ski for \$8100. The depreciated value y after t years is

$$y = 8100 - 929t, \qquad 0 \le t \le 6.$$

Sketch the graph of the equation.

65. ***Geometry*** A rectangle of length x and width w has a perimeter of 12 meters.

(a) Draw a rectangle that gives a visual representation of the problem. Use the specified variables to label the sides of the rectangle.

(b) Show that the width of the rectangle is $w = 6 - x$ and its area is $A = x(6 - x)$.

(c) Use a graphing utility to graph the area equation.

(d) From the graph in part (c), estimate the dimensions of the rectangle that yield a maximum area.

66. ***Geometry*** A rectangle of length x and width w has a perimeter of 22 yards.

(a) Draw a rectangle that gives a visual representation of the problem. Use the specified variables to label the sides of the rectangle.

(b) Show that the width of the rectangle is $w = 11 - x$ and its area is $A = x(11 - x)$.

(c) Use a graphing utility to graph the area equation. Be sure to adjust your window settings.

(d) From the graph in part (c), estimate the dimensions of the rectangle that yield a maximum area.

The symbol indicates an exercise or parts of an exercise in which you are instructed to use a graphing utility.

▶ Model It

67. ***Population Statistics*** The table shows the life expectancy of a child (at birth) in the United States for selected years from 1920 to 2000. (Source: U.S. National Center for Health Statistics, U.S. Census Bureau)

Year, t	Life expectancy, y
1920	54.1
1930	59.7
1940	62.9
1950	68.2
1960	69.7
1970	70.8
1980	73.7
1990	75.4
2000	77.1

A model for the life expectancy during this period is

$$y = -0.0025t^2 + 0.572t + 44.31$$

where y represents the life expectancy and t is the time in years, with $t = 20$ corresponding to 1920.

(a) Sketch a scatter plot of the data.

(b) Graph the model for the data and compare the scatter plot and the graph.

(c) Use the graph of the model to estimate the life expectancy of a child for the years 2005 and 2010.

(d) Do you think this model can be used to predict the life expectancy of a child 50 years from now? Explain.

68. ***Federal Debt*** The per capita federal debt y (in dollars) of the United States from 1950 through 2000 can be approximated by the model

$$y = 0.047t^3 + 9.23t^2 - 206.3t + 1984$$

where t is the time in years, with $t = 0$ corresponding to 1950. Use a graphing utility to graph the model and estimate the per capita federal debt for the year 2005. (Source: U. S. Census Bureau and U.S. Department of the Treasury)

69. ***Electronics*** The resistance y (in ohms) of 1000 feet of solid copper wire at 68 degrees Fahrenheit can be approximated by the model

$$y = \frac{10{,}770}{x^2} - 0.37, \qquad 5 \le x \le 100$$

where x is the diameter of the wire in mils (0.001 in.). Use a graphing utility to graph the model and estimate the resistance when $x = 50$. (Source: American Wire Gage)

Synthesis

True or False? **In Exercises 70 and 71, determine whether the statement is true or false. Justify your answer.**

70. In order to find the y-intercepts of the graph of an equation, let $y = 0$ and solve the equation for x.

71. The graph of a linear equation of the form $y = mx + b$ has one y-intercept.

72. ***Think About It*** Suppose you correctly enter an expression for the variable y on a graphing utility. However, no graph appears on the display when you graph the equation. Give a possible explanation and the steps you could take to remedy the problem. Illustrate your explanation with an example.

73. ***Think About It*** Find a and b if the graph of $y = ax^2 + bx^3$ is symmetric with respect to (a) the y-axis and (b) the origin. (There are many correct answers.)

74. In your own words, explain how the display of a graphing utility changes if the maximum setting for x is changed from 10 to 20.

Review

75. Identify the terms: $9x^5 + 4x^3 - 7$.

76. Rewrite the expression using exponential notation.

$-(7 \times 7 \times 7 \times 7)$

In Exercises 77–82, simplify the expression.

77. $\sqrt{18x} - \sqrt{2x}$

78. $\sqrt[4]{x^5}$

79. $\dfrac{70}{\sqrt{7x}}$

80. $\dfrac{55}{\sqrt{20} - 3}$

81. $\sqrt[6]{t^2}$

82. $\sqrt[3]{\sqrt{y}}$

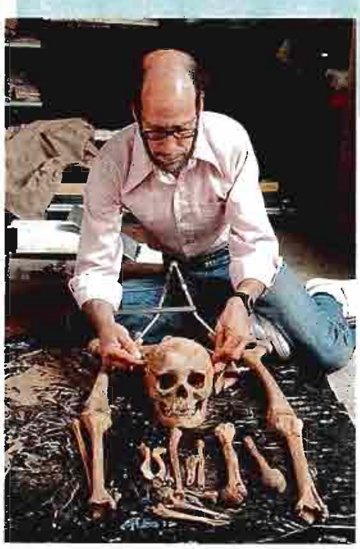
1.2 Linear Equations in One Variable

▶ What you should learn

- How to identify different types of equations
- How to solve linear equations in one variable
- How to solve equations that lead to linear equations
- How to find x- and y-intercepts of graphs of equations algebraically.
- How to use linear equations to model and solve real-life problems

▶ Why you should learn it

Linear equations are used in many real-life applications. For example, in Exercise 95 on page 95, linear equations can be used to model the relationship between the length of a thigh bone and the height of a person, helping researchers learn about ancient cultures.



M. Greenlar/The Image Works

Equations and Solutions of Equations

An **equation** in x is a statement that two algebraic expressions are equal. For example

$$3x - 5 = 7,\ x^2 - x - 6 = 0, \text{ and } \sqrt{2x} = 4$$

are equations. To **solve** an equation in x means to find all values of x for which the equation is true. Such values are **solutions.** For instance, $x = 4$ is a solution of the equation

$$3x - 5 = 7$$

because $3(4) - 5 = 7$ is a true statement.

The solutions of an equation depend on the kinds of numbers being considered. For instance, in the set of rational numbers, $x^2 = 10$ has no solution because there is no rational number whose square is 10. However, in the set of real numbers, the equation has the two solutions $\sqrt{10}$ and $-\sqrt{10}$.

An equation that is true for *every* real number in the domain of the variable is called an **identity.** For example

$$x^2 - 9 = (x + 3)(x - 3) \qquad \text{Identity}$$

is an identity because it is a true statement for any real value of x, and

$$\frac{x}{3x^2} = \frac{1}{3x} \qquad \text{Identity}$$

where $x \neq 0$, is an identity because it is true for any nonzero real value of x.

An equation that is true for just *some* (or even none) of the real numbers in the domain of the variable is called a **conditional equation.** For example, the equation

$$x^2 - 9 = 0 \qquad \text{Conditional equation}$$

is conditional because $x = 3$ and $x = -3$ are the only values in the domain that satisfy the equation. The equation $2x - 4 = 2x + 1$ is conditional because there are no real values of x for which the equation is true. Learning to solve conditional equations is the primary focus of this chapter.

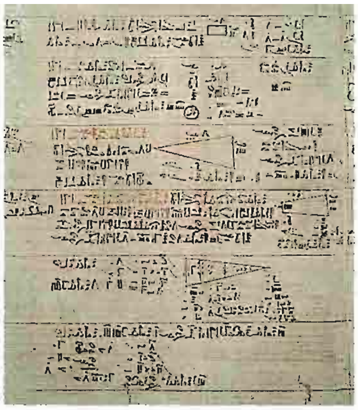
Linear Equations in One Variable

Definition of a Linear Equation

A **linear equation in one variable** x is an equation that can be written in the standard form

$$ax + b = 0$$

where a and b are real numbers with $a \neq 0$.



British Museum

Historical Note

This ancient Egyptian papyrus, discovered in 1858, contains one of the earliest examples of mathematical writing in existence. The papyrus itself dates back to around 1650 B.C., but it is actually a copy of writings from two centuries earlier. The algebraic equations on the papyrus were written in words. Diophantus, a Greek who lived around A.D. 250, is often called the Father of Algebra. He was the first to use abbreviated word forms in equations.

A linear equation has exactly one solution. To see this, consider the following steps. (Remember that $a \neq 0$.)

$ax + b = 0$	Write original equation.
$ax = -b$	Subtract b from each side.
$x = -\dfrac{b}{a}$	Divide each side by a.

To solve a conditional equation in x, isolate x on one side of the equation by a sequence of **equivalent** (and usually simpler) **equations,** each having the same solution(s) as the original equation. The operations that yield equivalent equations come from the Substitution Principle and the Properties of Equality studied in Chapter P.

Generating Equivalent Equations

An equation can be transformed into an *equivalent equation* by one or more of the following steps.

	Given Equation	*Equivalent Equation*
1. Remove symbols of grouping, combine like terms, or simplify fractions on one or both sides of the equation.	$2x - x = 4$	$x = 4$
2. Add (or subtract) the same quantity to (from) *each* side of the equation.	$x + 1 = 6$	$x = 5$
3. Multiply (or divide) *each* side of the equation by the same *nonzero* quantity.	$2x = 6$	$x = 3$
4. Interchange the two sides of the equation.	$2 = x$	$x = 2$

Exploration

Use a graphing utility to graph the equation $y = 3x - 6$. Use the result to estimate the x-intercept of the graph. Explain how the x-intercept is related to the solution of the equation $3x - 6 = 0$, as shown in Example 1(a).

Example 1 ▶ Solving a Linear Equation

a. $3x - 6 = 0$	Original equation
$3x = 6$	Add 6 to each side.
$x = 2$	Divide each side by 3.
b. $5x + 4 = 3x - 8$	Original equation
$2x + 4 = -8$	Subtract $3x$ from each side.
$2x = -12$	Subtract 4 from each side.
$x = -6$	Divide each side by 2.

After solving an equation, you should check each solution in the original equation. For instance, you can check the solution to Example 1(a) as follows.

$$3x - 6 = 0 \qquad \text{Write original equation.}$$

$$3(2) - 6 \stackrel{?}{=} 0 \qquad \text{Substitute 2 for } x.$$

$$0 = 0 \qquad \text{Solution checks. } \checkmark$$

Try checking the solution to Example 1(b).

Some linear equations have no solutions because all the x-terms sum to zero and a contradictory (false) statement such as $0 = 5$ or $12 = 7$ is obtained. For instance, the linear equation

$$x = x + 1$$

has no solution. Watch for this type of linear equation in the exercises.

Example 2 ▶ Solving a Linear Equation

Solve

$$6(x - 1) + 4 = 3(7x + 1).$$

Solution

$6(x - 1) + 4 = 3(7x + 1)$	Write original equation.
$6x - 6 + 4 = 21x + 3$	Distributive Property
$6x - 2 = 21x + 3$	Simplify.
$6x = 21x + 5$	Add 2 to each side.
$-15x = 5$	Subtract $21x$ from each side.
$x = -\dfrac{1}{3}$	Divide each side by -15.

Check

Check this solution by substituting $-\frac{1}{3}$ for x in the original equation.

$6(x - 1) + 4 = 3(7x + 1)$	Write original equation.
$6\left(-\frac{1}{3} - 1\right) + 4 \stackrel{?}{=} 3\left[7\left(-\frac{1}{3}\right) + 1\right]$	Substitute $-\frac{1}{3}$ for x.
$6\left(-\frac{4}{3}\right) + 4 \stackrel{?}{=} 3\left[-\frac{7}{3} + 1\right]$	Simplify.
$6\left(-\frac{4}{3}\right) + 4 \stackrel{?}{=} 3\left(-\frac{4}{3}\right)$	Simplify.
$-\frac{24}{3} + 4 \stackrel{?}{=} -\frac{12}{3}$	Multiply.
$-8 + 4 \stackrel{?}{=} -4$	Simplify.
$-4 = -4$	Solution checks. ✓

So, the solution is $x = -\frac{1}{3}$.

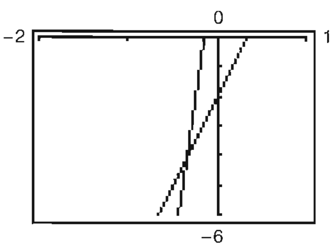
Technology

You can use a graphing utility to check that a solution is reasonable. One way to do this is to graph the left side of the equation, then graph the right side of the equation, and determine the point of intersection. For instance, in Example 2, if you graph the equations

$y_1 = 6(x - 1) + 4$ The left side

$y_2 = 3(7x + 1)$ The right side

in the same viewing window, they should intersect when $x = -\frac{1}{3}$, as shown in the graph below.



Equations That Lead to Linear Equations

To solve an equation involving fractional expressions, find the least common denominator (LCD) of all terms and multiply every term by the LCD.

Example 3 ▶ An Equation Involving Fractional Expressions

Solve $\frac{x}{3} + \frac{3x}{4} = 2$.

Solution

$$\frac{x}{3} + \frac{3x}{4} = 2 \qquad \text{Write original equation.}$$

$$(12)\frac{x}{3} + (12)\frac{3x}{4} = (12)2 \qquad \text{Multiply each term by the LCD of 12.}$$

$$4x + 9x = 24 \qquad \text{Divide out and multiply.}$$

$$13x = 24 \qquad \text{Combine like terms.}$$

$$x = \frac{24}{13} \qquad \text{Divide each side by 13.}$$

The solution is $x = \frac{24}{13}$. Check this in the original equation.

STUDY TIP

An equation with a *single fraction* on each side can be cleared of denominators by **cross multiplying,** which is equivalent to multiplying by the LCD and then dividing out.

$$\frac{a}{b} = \frac{c}{d} \qquad \text{LCD is } bd.$$

$$\frac{a}{b} \cdot bd = \frac{c}{d} \cdot bd \qquad \text{Multiply by LCD.}$$

$$ad = cb \qquad \text{Divide out common factors.}$$

By comparing the last equation with the original equation, you can see that the left numerator was multiplied by the right denominator and the right numerator was multiplied by the left denominator. Try cross multiplying the following equation to clear the equation of denominators.

$$\frac{2}{x-3} = \frac{3}{x+1}$$

When multiplying or dividing an equation by a *variable* quantity, it is possible to introduce an extraneous solution. An **extraneous solution** is one that does not satisfy the original equation.

Example 4 ▶ An Equation with an Extraneous Solution

Solve $\frac{1}{x-2} = \frac{3}{x+2} - \frac{6x}{x^2-4}$.

Solution

The LCD is $x^2 - 4$, or $(x + 2)(x - 2)$. Multiply each term by this LCD.

$$\frac{1}{x-2}(x+2)(x-2) = \frac{3}{x+2}(x+2)(x-2) - \frac{6x}{x^2-4}(x+2)(x-2)$$

$$x + 2 = 3(x - 2) - 6x, \qquad x \neq \pm 2$$

$$x + 2 = 3x - 6 - 6x$$

$$x + 2 = -3x - 6$$

$$4x = -8$$

$$x = -2 \qquad \text{Extraneous solution}$$

In the original equation, $x = -2$ yields a denominator of zero. So, $x = -2$ is an extraneous solution, and the original equation has *no solution*.

Finding Intercepts Algebraically

In Section 1.1 you learned to find x- and y-intercepts using a graphical approach. You can also use an algebraic approach to find x- and y-intercepts, as follows.

> **Finding Intercepts Algebraically**
>
> 1. To find x-intercepts, set y equal to zero and solve the equation for x.
> 2. To find y-intercepts, set x equal to zero and solve the equation for y.

Here is an example.

$$y = 4x + 1 \Longrightarrow 0 = 4x + 1 \Longrightarrow -1 = 4x \Longrightarrow -\tfrac{1}{4} = x$$

$$y = 4x + 1 \Longrightarrow y = 4(0) + 1 \Longrightarrow y = 1$$

So, the x-intercept of $y = 4x + 1$ is $\left(-\frac{1}{4}, 0\right)$ and the y-intercept is $(0, 1)$.

Application

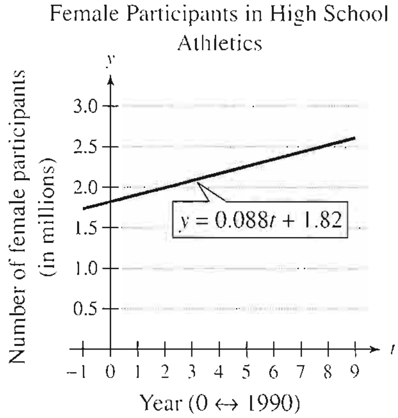


FIGURE 1.14

Example 5 ▶ Female Participants in Athletic Programs

The number y of female participants in high school athletic programs (in millions) in the United States from 1989 to 1999 can be approximated by the linear model

$$y = 0.088t + 1.82, \qquad -1 \le t \le 9$$

where $t = 0$ represents 1990. (a) Find the y-intercept of the graph of the linear model shown in Figure 1.14 algebraically. (b) Assuming that this linear pattern continues, find the year in which there will be 3.14 million female participants. (Source: National Federation of State High School Associations)

Solution

a. To find the y-intercept, let $t = 0$ and solve for y as follows.

$y = 0.088t + 1.82$ — Write original equation.

$= 0.088(0) + 1.82$ — Substitute 0 for t.

$= 1.82$ — Simplify.

So, the y-intercept is $(0, 1.82)$.

b. Let $y = 3.14$ and solve the equation $3.14 = 0.088t + 1.82$ for t.

$3.14 = 0.088t + 1.82$ — Write original equation.

$1.32 = 0.088t$ — Subtract 1.82 from each side.

$15 = t$ — Divide each side by 0.088.

Because $t = 0$ represents 1990, $t = 15$ must represent 2005. So, from this model, there will be 3.14 million female participants in 2005.

1.2 Exercises

In Exercises 1–10, determine whether each value of x is a solution of the equation.

Equation	Values	
1. $5x - 3 = 3x + 5$	(a) $x = 0$	(b) $x = -5$
	(c) $x = 4$	(d) $x = 10$
2. $7 - 3x = 5x - 17$	(a) $x = -3$	(b) $x = 0$
	(c) $x = 8$	(d) $x = 3$
3. $3x^2 + 2x - 5$	(a) $x = -3$	(b) $x = 1$
$= 2x^2 - 2$	(c) $x = 4$	(d) $x = -5$
4. $5x^3 + 2x - 3$	(a) $x = 2$	(b) $x = -2$
$= 4x^3 + 2x - 11$	(c) $x = 0$	(d) $x = 10$
5. $\frac{5}{2x} - \frac{4}{x} = 3$	(a) $x = -\frac{1}{2}$	(b) $x = 4$
	(c) $x = 0$	(d) $x = \frac{1}{4}$
6. $3 + \frac{1}{x+2} = 4$	(a) $x = -1$	(b) $x = -2$
	(c) $x = 0$	(d) $x = 5$
7. $\sqrt{3x - 2} = 4$	(a) $x = 3$	(b) $x = 2$
	(c) $x = 9$	(d) $x = -6$
8. $\sqrt[3]{x - 8} = 3$	(a) $x = 2$	(b) $x = -5$
	(c) $x = 35$	(d) $x = 8$
9. $6x^2 - 11x - 35 = 0$	(a) $x = -\frac{5}{3}$	(b) $x = -\frac{2}{7}$
	(c) $x = \frac{7}{2}$	(d) $x = \frac{5}{3}$
10. $10x^2 + 21x - 10 = 0$	(a) $x = \frac{2}{5}$	(b) $x = -\frac{5}{2}$
	(c) $x = -\frac{1}{3}$	(d) $x = -2$

In Exercises 11–20, determine whether the equation is an identity or a conditional equation.

11. $2(x - 1) = 2x - 2$

12. $3(x + 2) = 5x + 4$

13. $-6(x - 3) + 5 = -2x + 10$

14. $3(x + 2) - 5 = 3x + 1$

15. $4(x + 1) - 2x = 2(x + 2)$

16. $-7(x - 3) + 4x = 3(7 - x)$

17. $x^2 - 8x + 5 = (x - 4)^2 - 11$

18. $x^2 + 2(3x - 2) = x^2 + 6x - 4$

19. $3 + \frac{1}{x + 1} = \frac{4x}{x + 1}$

20. $\frac{5}{x} + \frac{3}{x} = 24$

In Exercises 21 and 22, justify each step of the solution.

21.
$$\begin{aligned} 4x + 32 &= 83 \\ 4x + 32 - 32 &= 83 - 32 \\ 4x &= 51 \\ \frac{4x}{4} &= \frac{51}{4} \\ x &= \frac{51}{4} \end{aligned}$$

22.
$$\begin{aligned} 3(x - 4) + 10 &= 7 \\ 3x - 12 + 10 &= 7 \\ 3x - 2 &= 7 \\ 3x - 2 + 2 &= 7 + 2 \\ 3x &= 9 \\ \frac{3x}{3} &= \frac{9}{3} \\ x &= 3 \end{aligned}$$

In Exercises 23–38, solve the equation and check your solution.

23. $x + 11 = 15$

24. $7 - x = 19$

25. $7 - 2x = 25$

26. $7x + 2 = 23$

27. $8x - 5 = 3x + 20$

28. $7x + 3 = 3x - 17$

29. $2(x + 5) - 7 = 3(x - 2)$

30. $3(x + 3) = 5(1 - x) - 1$

31. $x - 3(2x + 3) = 8 - 5x$

32. $9x - 10 = 5x + 2(2x - 5)$

33. $\frac{5x}{4} + \frac{1}{2} = x - \frac{1}{2}$

34. $\frac{x}{5} - \frac{x}{2} = 3 + \frac{3x}{10}$

35. $\frac{3}{2}(z + 5) - \frac{1}{4}(z + 24) = 0$

36. $\frac{3x}{2} + \frac{1}{4}(x - 2) = 10$

37. $0.25x + 0.75(10 - x) = 3$

38. $0.60x + 0.40(100 - x) = 50$

In Exercises 39–42, solve the equation in two ways. Then explain which way is easier.

39. $3(x - 1) = 4$

40. $4(x + 3) = 15$

41. $\frac{1}{3}(x + 2) = 5$

42. $\frac{3}{4}(z - 4) = 6$

Graphical Analysis **In Exercises 43–48, use a graphing utility to graph the equation and approximate any x-intercepts. Set $y = 0$ and solve the resulting equation. Compare the results with the graph's x-intercepts.**

43. $y = 2(x - 1) - 4$

44. $y = \frac{4}{3}x + 2$

45. $y = 20 - (3x - 10)$

46. $y = 10 + 2(x - 2)$

47. $y = -38 + 5(9 - x)$

48. $y = 6x - 6\left(\frac{16}{11} + x\right)$

In Exercises 49–58, find the x- and y-intercepts of the graph of the equation algebraically.

49. $y = 12 - 5x$

50. $y = 16 - 3x$

51. $y = -3(2x + 1)$

52. $y = 5 - (6 - x)$

53. $2x + 3y = 10$

54. $4x - 5y = 12$

55. $\dfrac{2x}{5} + 8 - 3y = 0$

56. $\dfrac{8x}{3} + 5 - 2y = 0$

57. $4y - 0.75x + 1.2 = 0$

58. $3y + 2.5x - 3.4 = 0$

In Exercises 59–80, solve the equation and check your solution. (If not possible, explain why.)

59. $x + 8 = 2(x - 2) - x$

60. $8(x + 2) - 3(2x + 1) = 2(x + 5)$

61. $\dfrac{100 - 4x}{3} = \dfrac{5x + 6}{4} + 6$

62. $\dfrac{17 + y}{y} + \dfrac{32 + y}{y} = 100$

63. $\dfrac{5x - 4}{5x + 4} = \dfrac{2}{3}$

64. $\dfrac{10x + 3}{5x + 6} = \dfrac{1}{2}$

65. $10 - \dfrac{13}{x} = 4 + \dfrac{5}{x}$

66. $\dfrac{15}{x} - 4 = \dfrac{6}{x} + 3$

67. $3 = 2 + \dfrac{2}{z + 2}$

68. $\dfrac{1}{x} + \dfrac{2}{x - 5} = 0$

69. $\dfrac{x}{x + 4} + \dfrac{4}{x + 4} + 2 = 0$

70. $\dfrac{7}{2x + 1} - \dfrac{8x}{2x - 1} = -4$

71. $\dfrac{2}{(x - 4)(x - 2)} = \dfrac{1}{x - 4} + \dfrac{2}{x - 2}$

72. $\dfrac{4}{x - 1} + \dfrac{6}{3x + 1} = \dfrac{15}{3x + 1}$

73. $\dfrac{1}{x - 3} + \dfrac{1}{x + 3} = \dfrac{10}{x^2 - 9}$

74. $\dfrac{1}{x - 2} + \dfrac{3}{x + 3} = \dfrac{4}{x^2 + x - 6}$

75. $\dfrac{3}{x^2 - 3x} + \dfrac{4}{x} = \dfrac{1}{x - 3}$

76. $\dfrac{6}{x} - \dfrac{2}{x + 3} = \dfrac{3(x + 5)}{x^2 + 3x}$

77. $(x + 2)^2 + 5 = (x + 3)^2$

78. $(x + 1)^2 + 2(x - 2) = (x + 1)(x - 2)$

79. $(x + 2)^2 - x^2 = 4(x + 1)$

80. $(2x + 1)^2 = 4(x^2 + x + 1)$

In Exercises 81–88, solve for x.

81. $4(x + 1) - ax = x + 5$

82. $4 - 2(x - 2b) = ax + 3$

83. $6x + ax = 2x + 5$

84. $5 + ax = 12 - bx$

85. $19x + \frac{1}{2}ax = x + 9$

86. $-5(3x - 6b) + 12 = 8 + 3ax$

87. $-2ax + 6(x + 3) = -4x + 1$

88. $\frac{4}{5}x - ax = 2\left(\frac{2}{5}x - 1\right) + 10$

In Exercises 89–92, solve the equation for x. (Round your solution to three decimal places.)

89. $0.275x + 0.725(500 - x) = 300$

90. $2.763 - 4.5(2.1x - 5.1432) = 6.32x + 5$

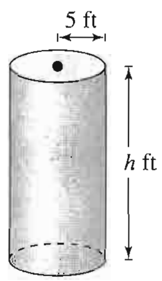
91. $\dfrac{2}{7.398} - \dfrac{4.405}{x} = \dfrac{1}{x}$

92. $\dfrac{3}{6.350} - \dfrac{6}{x} = 18$

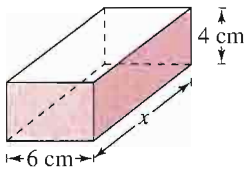
93. ***Geometry*** The surface area S of the circular cylinder shown in the figure is

$$S = 2\pi(25) + 2\pi(5h)$$

Find the height h of the cylinder if the surface area is 471 square feet. Use 3.14 for π.



94. ***Geometry*** The surface area S of the rectangular solid in the figure is $S = 2(24) + 2(4x) + 2(6x)$. Find the length x of the box if the surface area is 248 square centimeters.



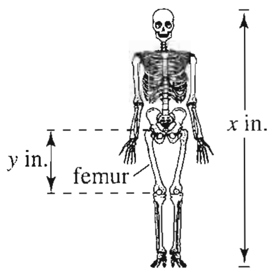
▶ Model It

95. ***Anthropology*** The relationship between the length of an adult's femur (thigh bone) and the height of the adult can be approximated by the linear equations

$y = 0.432x - 10.44$ Female

$y = 0.449x - 12.15$ Male

where y is the length of the femur in inches and x is the height of the adult in inches (see figure).



(a) An anthropologist discovers a femur belonging to an adult human female. The bone is 16 inches long. Estimate the height of the female.

(b) From the foot bones of an adult human male, an anthropologist estimates that the person's height was 69 inches. A few feet away from the site where the foot bones were discovered, the anthropologist discovers a male adult femur that is 19 inches long. Is it likely that both bones came from the same person?

(c) Complete the table to determine if there is a height of an adult for which an anthropologist would not be able to determine whether the femur belonged to a male or a female.

▶ Model It *(continued)*

Height, x	Female femur length, y	Male femur length, y
60		
70		
80		
90		
100		
110		

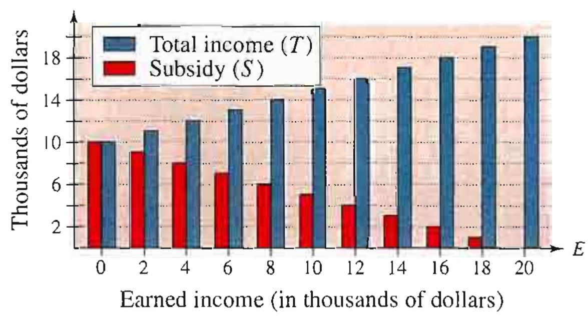
(d) Solve part (c) algebraically by setting the two equations equal to each other and solving for x. Compare your solutions. Do you believe an anthropologist would ever have the problem of not being able to determine whether a femur belonged to a male or female? Why or why not?

96. ***Tax Credits*** Use the following information about a possible tax credit for a family consisting of two adults and two children (see figure).

Earned income: E

Subsidy: $S = 10{,}000 - \frac{1}{2}E, \quad 0 \le E \le 20{,}000$

Total income: $T = E + S$



(a) Express the total income T in terms of E.

(b) Find the earned income E if the subsidy is \$6600.

(c) Find the earned income E if the total income is \$13,800.

(d) Find the subsidy S if the total income is \$12,500.

97. ***Consumerism*** The number of light trucks sold y (in millions) in the United States from 1992 to 1999 can be approximated by the model

$$y = 0.451t + 3.81$$

where $t = 2$ represents 1992. (Source: U.S. Bureau of Economic Analysis)

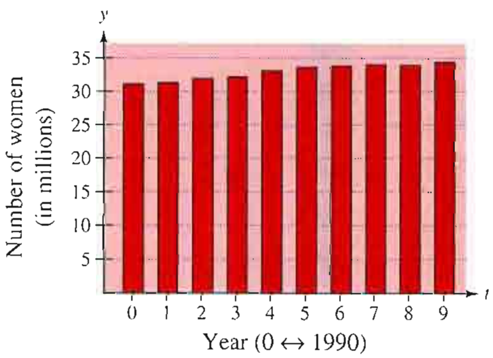
(a) Use the model to create a line graph of the number of light trucks sold from 1992 to 1999.

(b) Use the graph to determine the year during which the number of light trucks sold reached 6 million.

98. ***Labor Statistics*** The number of married women y (in millions) in the civilian work force in the United States from 1990 to 1999 (see figure) can be approximated by the model

$$y = 0.41t + 30.9$$

where $t = 0$ represents 1990. According to this model, during which year did the number reach 33 million? Explain how to answer this question graphically and algebraically. (Source: U.S. Bureau of Labor Statistics)



99. ***Operating Cost*** A delivery company has a fleet of vans. The annual operating cost C per van is

$$C = 0.32m + 2500$$

where m is the number of miles traveled by a van in a year. What number of miles will yield an annual operating cost of \$10,000?

100. ***Flood Control*** A river has risen 8 feet above its flood stage. The water begins to recede at a rate of 3 inches per hour. Write a mathematical model that shows the number of feet above flood stage after t hours. If the water continually recedes at this rate, when will the river be 1 foot above its flood stage?

Synthesis

True or False? **In Exercises 101 and 102, determine whether the statement is true or false. Justify your answer.**

101. The equation $x(3 - x) = 10$ is a linear equation.

102. The equation $x^2 + 9x - 5 = 4 - x^3$ has no real solution.

103. ***Think About It*** What is meant by "equivalent equations"? Give an example of two equivalent equations.

104. ***Writing*** Describe the steps used to transform an equation into an equivalent equation.

105. ***Exploration***

(a) Complete the table.

x	-1	0	1	2	3	4
$3.2x - 5.8$						

(b) Use the table in part (a) to determine the interval in which the solution to the equation $3.2x - 5.8 = 0$ is located. Explain your reasoning.

(c) Complete the table.

x	1.5	1.6	1.7	1.8	1.9	2.0
$3.2x - 5.8$						

(d) Use the table in part (c) to determine the interval in which the solution to the equation $3.2x - 5.8 = 0$ is located. Explain how this process can be used to approximate the solution to any desired degree of accuracy.

106. ***Exploration*** Use the procedure in Exercise 105 to approximate the solution to the equation $0.3(x - 1.5) - 2 = 0$ accurate to two decimal places.

Review

In Exercises 107 and 108, simplify the expression.

107. $\dfrac{x^2 + 5x - 36}{2x^2 + 17x - 9}$

108. $\dfrac{x^2 - 49}{x^3 + x^2 + 3x - 21}$

In Exercises 109–112, sketch the graph of the equation.

109. $y = 3x - 5$

110. $y = -\frac{1}{2}x - \frac{9}{2}$

111. $y = -x^2 - 5x$

112. $y = \sqrt{5 - x}$

1.3 Modeling with Linear Equations

▶ **What you should learn**

- How to use a verbal model in a problem-solving plan
- How to write and use mathematical models to solve real-life problems
- How to solve mixture problems
- How to use common formulas to solve real-life problems

▶ **Why you should learn it**

You can use linear equations to determine the percents of incomes and of expenses of the federal government that come from various sources. See Exercise 44 on page 105.

Bob Rowan; Progressive Image/Corbis

Introduction to Problem Solving

In this section you will learn how algebra can be used to solve problems that occur in real-life situations. The process of translating phrases or sentences into algebraic expressions or equations is called **mathematical modeling.** A good approach to mathematical modeling is to use two stages. Begin by using the verbal description of the problem to form a *verbal model*. Then, after assigning labels to the quantities in the verbal model, form a *mathematical model* or *algebraic equation*.

Verbal Description ⇒ Verbal Model ⇒ Algebraic Equation

When you are trying to construct a verbal model, it is helpful to look for a *hidden equality*—a statement that two algebraic expressions are equal.

Example 1 ▶ Using a Verbal Model

You have accepted a job for which your annual salary will be $27,236. This salary includes a year-end bonus of $500. You will be paid twice a month. What will your gross pay be for each paycheck?

Solution

Because there are 12 months in a year and you will be paid twice a month, it follows that you will receive 24 paychecks during the year. You can construct an algebraic equation for this problem as follows. Begin with a verbal model, then assign labels, and finally form an algebraic equation.

Verbal Model: Income for year = 24 paychecks + Bonus

Labels:
Income for year = 27,236 (dollars)
Amount of each paycheck = x (dollars)
Bonus = 500 (dollars)

Equation: $27{,}236 = 24x + 500$

The algebraic equation for this problem is a *linear equation* in the variable x, which you can solve as follows.

$$27{,}236 = 24x + 500 \qquad \text{Write original equation.}$$

$$27{,}236 - 500 = 24x + 500 - 500 \qquad \text{Subtract 500 from each side.}$$

$$26{,}736 = 24x \qquad \text{Simplify.}$$

$$\frac{26{,}736}{24} = \frac{24x}{24} \qquad \text{Divide each side by 24.}$$

$$1114 = x \qquad \text{Simplify.}$$

So, your gross pay for each paycheck will be $1114.

A fundamental step in writing a mathematical model to represent a real-life problem is translating key words and phrases into algebraic expressions and equations. The following list gives several examples.

Translating Key Words and Phrases

Key Words and Phrases	*Verbal Description*	*Algebraic Expression or Equation*
Equality:		
Equals, equal to, is, are, was, will be, represents	• The sale price S is \$10 less than the list price L.	$S = L - 10$
Addition:		
Sum, plus, greater than, increased by, more than, exceeds, total of	• The sum of 5 and x • Seven more than y	$5 + x$ or $x + 5$ $7 + y$ or $y + 7$
Subtraction:		
Difference, minus, less than, decreased by, subtracted from, reduced by, the remainder	• The difference of 4 and b • Three less than z	$4 - b$ $z - 3$
Multiplication:		
Product, multiplied by, twice, times, percent of	• Two times x • Three percent of t	$2x$ $0.03t$
Division:		
Quotient, divided by, ratio, per	• The ratio of x to 8	$\frac{x}{8}$

Using Mathematical Models

Example 2 ▶ Finding the Percent of a Raise

You have accepted a job that pays \$8 an hour. You are told that after a two-month probationary period, your hourly wage will be increased to \$9 an hour. What percent raise will you receive after the two-month period?

Solution

Verbal Model: Raise = Percent · Old wage

Labels:
Old wage = 8 (dollars per hour)
New wage = 9 (dollars per hour)
Raise = 9 − 8 = 1 (dollars per hour)
Percent = r (percent in decimal form)

Equation:

$$1 = r \cdot 8$$

$$\frac{1}{8} = r \quad \text{Divide each side by 8.}$$

$$0.125 = r \quad \text{Rewrite fraction as a decimal.}$$

You will receive a raise of 0.125 or 12.5%.

STUDY TIP

Writing the unit for each label in a real-life problem helps you determine the unit for the answer. This is called *unit analysis*. When the same unit of measure occurs in the numerator and denominator of an expression, you can divide out the unit. For instance, unit analysis verifies that the unit for time in the formula below is hours.

$$\text{Time} = \frac{\text{distance}}{\text{rate}}$$

$$= \frac{\text{miles}}{\frac{\text{miles}}{\text{hour}}}$$

$$= \cancel{\text{miles}} \cdot \frac{\text{hours}}{\cancel{\text{miles}}}$$

$$= \text{hours}$$

Example 3 ▶ Finding the Percent of Monthly Expenses

Your family has an annual income of \$57,000 and the following monthly expenses: mortgage (\$1100), car payment (\$375), food (\$295), utilities (\$240), and credit cards (\$220). The total value of the monthly expenses represents what percent of your family's annual income?

Solution

The total amount of your family's monthly expenses is \$2230. The total monthly expenses for 1 year are \$26,760.

Verbal Model: Monthly expenses = Percent · Income

Labels:
Income = 57,000 (dollars)
Monthly expenses = 26,760 (dollars)
Percent = r (in decimal form)

Equation: $26{,}760 = r \cdot 57{,}000$

$\frac{26{,}760}{57{,}000} = r$ Divide each side by 57,000.

$0.469 \approx r$ Use a calculator.

Your family's monthly expenses are approximately 0.469 or 46.9% of your family's annual income.

Example 4 ▶ Finding the Dimensions of a Room

A rectangular kitchen is twice as long as it is wide, and its perimeter is 84 feet. Find the dimensions of the kitchen.

Solution

For this problem, it helps to sketch a diagram, as shown in Figure 1.15.

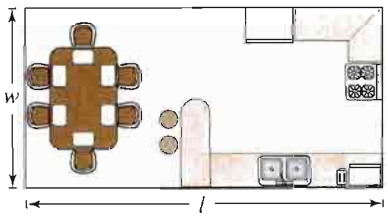


FIGURE 1.15

Verbal Model: 2 · Length + 2 · Width = Perimeter

Labels:
Perimeter = 84 (feet)
Width = w (feet)
Length = $l = 2w$ (feet)

Equation: $2(2w) + 2w = 84$

$6w = 84$ Group like terms.

$w = 14$ Divide each side by 6.

Because the length is twice the width, you have

$l = 2w$ Length is twice width.

$= 2(14)$ Substitute.

$= 28.$ Simplify.

So, the dimensions of the room are 14 feet by 28 feet.

Example 5 ▶ A Distance Problem

A plane is flying nonstop from Atlanta to Portland, a distance of about 2700 miles, as shown in Figure 1.16. After 1.5 hours in the air, the plane flies over Kansas City (a distance of 820 miles from Atlanta). Estimate the time it will take the plane to fly from Atlanta to Portland.

FIGURE 1.16

Solution

Verbal Model: Distance = Rate · Time

Labels:
Distance = 2700 (miles)
Time = t (hours)
$$\text{Rate} = \frac{\text{distance to Kansas City}}{\text{time to Kansas City}} = \frac{820}{1.5}$$ (miles per hour)

Equation:
$$2700 = \frac{820}{1.5}t$$
$$4050 = 820t$$
$$\frac{4050}{820} = t$$
$$4.94 \approx t$$

The trip will take about 4.94 hours, or about 4 hours and 57 minutes.

Example 6 ▶ An Application Involving Similar Triangles

To determine the height of the Aon Center Building (in Chicago), you measure the shadow cast by the building and find it to be 142 feet long, as shown in Figure 1.17. Then you measure the shadow cast by a four-foot post and find it to be 6 inches long. Estimate the building's height.

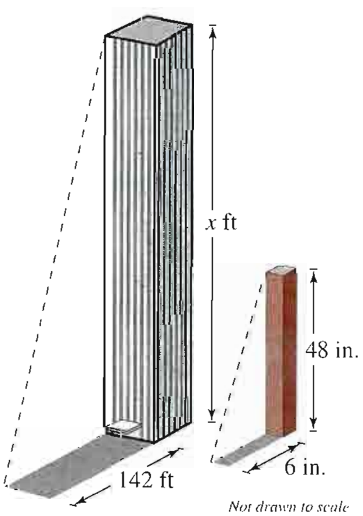


FIGURE 1.17

Solution

To solve this problem, you use a result from geometry that states that the ratios of corresponding sides of similar triangles are equal.

Verbal Model:
$$\frac{\text{Height of building}}{\text{Length of building's shadow}} = \frac{\text{Height of post}}{\text{Length of post's shadow}}$$

Labels:
Height of building = x (feet)
Length of building's shadow = 142 (feet)
Height of post = 4 feet = 48 inches (inches)
Length of post's shadow = 6 (inches)

Equation:
$$\frac{x}{142} = \frac{48}{6}$$
$$x = 1136$$

So, the Aon Center Building is about 1136 feet high.

Mixture Problems

Problems that involve two or more rates are called mixture problems. They are not limited to mixtures of chemical solutions, as shown in Examples 7 and 8.

Example 7 ▶ A Simple Interest Problem

You invested a total of \$10,000 at $4\frac{1}{2}\%$ and $5\frac{1}{2}\%$ simple interest. During 1 year, the two accounts earned \$508.75. How much did you invest in each?

STUDY TIP

Example 7 uses the simple interest formula $I = Prt$, where I is the interest, P is the principal, r is the annual interest rate (in decimal form), and t is the time in years.

Solution

Verbal Model: Interest from $4\frac{1}{2}\%$ + Interest from $5\frac{1}{2}\%$ = Total interest

Labels:
Amount invested at $4\frac{1}{2}\% = x$ (dollars)
Amount invested at $5\frac{1}{2}\% = 10{,}000 - x$ (dollars)
Interest from $4\frac{1}{2}\% = Prt = (x)(0.045)(1)$ (dollars)
Interest from $5\frac{1}{2}\% = Prt = (10{,}000 - x)(0.055)(1)$ (dollars)
Total interest $= 508.75$ (dollars)

Equation:
$$0.045x + 0.055(10{,}000 - x) = 508.75$$
$$-0.01x = -41.25$$
$$x = 4125$$

So, \$4125 was invested at $4\frac{1}{2}\%$ and $10{,}000 - x$ or \$5875 was invested at $5\frac{1}{2}\%$.

Example 8 ▶ An Inventory Problem

A store has \$30,000 of inventory in 13-inch and 19-inch color televisions. The profit on a 13-inch set is 22% and the profit on a 19-inch set is 40%. The profit for the entire stock is 35%. How much was invested in each type of television?

Solution

Verbal Model: Profit from 13-inch sets + Profit from 19-inch sets = Total profit

Labels:
Inventory of 13-inch sets $= x$ (dollars)
Inventory of 19-inch sets $= 30{,}000 - x$ (dollars)
Profit from 13-inch sets $= 0.22x$ (dollars)
Profit from 19-inch sets $= 0.40(30{,}000 - x)$ (dollars)
Total profit $= 0.35(30{,}000) = 10{,}500$ (dollars)

Equation:
$$0.22x + 0.40(30{,}000 - x) = 10{,}500$$
$$-0.18x = -1500$$
$$x \approx 8333.33$$

So, \$8333.33 was invested in 13-inch sets and $30{,}000 - x$ or \$21,666.67 was invested in 19-inch sets.

Common Formulas

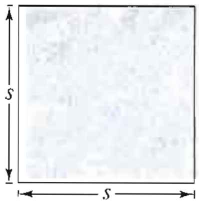
Many common types of geometric, scientific, and investment problems use ready-made equations called **formulas.** Knowing these formulas will help you translate and solve a wide variety of real-life applications.

Common Formulas for Area A, Perimeter P, Circumference C, and Volume V

Square

$A = s^2$

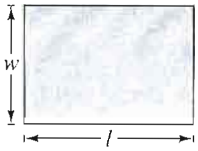
$P = 4s$



Rectangle

$A = lw$

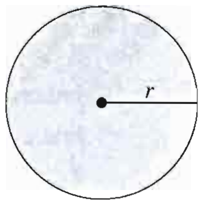
$P = 2l + 2w$



Circle

$A = \pi r^2$

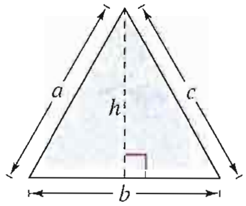
$C = 2\pi r$



Triangle

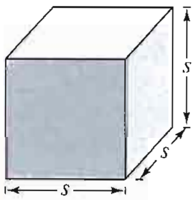
$A = \frac{1}{2}bh$

$P = a + b + c$



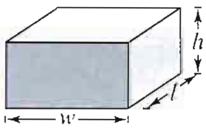
Cube

$V = s^3$



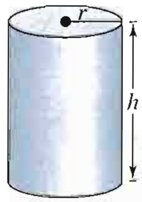
Rectangular Solid

$V = lwh$



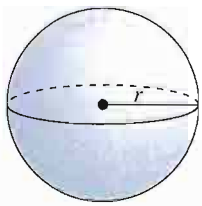
Circular Cylinder

$V = \pi r^2 h$



Sphere

$V = \frac{4}{3}\pi r^3$



Miscellaneous Common Formulas

Temperature:

$F = \frac{9}{5}C + 32$ — F = degrees Fahrenheit, C = degrees Celsius

Simple Interest:

$I = Prt$ — I = interest, P = principal, r = annual interest rate, t = time in years

Compound Interest:

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ — A = balance, P = principal, r = annual interest rate, n = compoundings per year, t = time in years

Distance:

$d = rt$ — d = distance traveled, r = rate, t = time

When working with applied problems, you often need to rewrite one of the common formulas. For instance, the formula for the perimeter of a rectangle, $P = 2l + 2w$, can be rewritten or solved for w as $w = \frac{1}{2}(P - 2l)$.

Example 9 ▶ Using a Formula

A cylindrical can has a volume of 200 cubic centimeters (cm^3) and a radius of 4 centimeters (cm), as shown in Figure 1.18. Find the height of the can.

FIGURE 1.18

Solution

The formula for the *volume of a cylinder* is $V = \pi r^2 h$. To find the height of the can, solve for h.

$$h = \frac{V}{\pi r^2}$$

Then, using $V = 200$ and $r = 4$, find the height.

$$h = \frac{200}{\pi(4)^2} \qquad \text{Substitute 200 for } V \text{ and 4 for } r.$$

$$= \frac{200}{16\pi} \qquad \text{Simplify denominator.}$$

$$\approx 3.98 \qquad \text{Use a calculator.}$$

You can use unit analysis to check that your answer is reasonable.

$$\frac{200\ \cancel{\text{cm}^3}}{16\pi\ \cancel{\text{cm}^2}} \approx 3.98\ \text{cm}$$

Writing ABOUT MATHEMATICS

Translating Algebraic Formulas Most people use algebraic formulas every day—sometimes without realizing it because they use a verbal form or think of an often-repeated calculation in steps. Translate each of the following verbal descriptions into an algebraic formula, and demonstrate the use of each formula.

a. ***Designing Billboards*** "The letters on a sign or billboard are designed to be readable at a certain distance. Take half the letter height in inches and multiply by 100 to find the readable distance in feet."—Thos. Hodgson, Hodgson Signs (Source: *Rules of Thumb* by Tom Parker)

b. ***Percent of Calories from Fat*** "To calculate percent of calories from fat, multiply grams of total fat per serving by 9, then divide by the number of calories per serving." (Source: *Good Housekeeping*)

c. ***Building Stairs*** "A set of steps will be comfortable to use if two times the height of one riser plus the width of one tread is equal to 26 inches."—Alice Lukens Bachelder, gardener (Source: *Rules of Thumb* by Tom Parker)

1.3 Exercises

In Exercises 1–10, write a verbal description of the algebraic expression without using the variable.

1. $x + 4$
2. $t - 10$
3. $\frac{u}{5}$
4. $\frac{2}{3}x$
5. $\frac{y - 4}{5}$
6. $\frac{z + 10}{7}$
7. $-3(b + 2)$
8. $\frac{-5(x - 1)}{8}$
9. $12x(x - 5)$
10. $\frac{(q + 4)(3 - q)}{2q}$

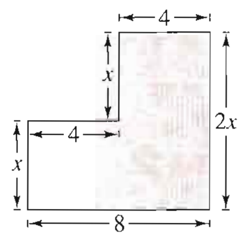
In Exercises 11–22, write an algebraic expression for the verbal description.

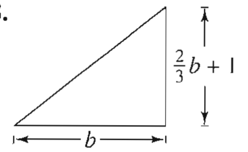
11. The sum of two consecutive natural numbers
12. The product of two consecutive natural numbers
13. The product of two consecutive odd integers, the first of which is $2n - 1$
14. The sum of the squares of two consecutive even integers, the first of which is $2n$
15. The distance traveled in t hours by a car traveling at 50 miles per hour
16. The travel time for a plane traveling at a rate of r kilometers per hour for 200 kilometers
17. The amount of acid in x liters of a 20% acid solution
18. The sale price of an item that is discounted 20% of its list price L
19. The perimeter of a rectangle with a width x and a length that is twice the width
20. The area of a triangle with base 20 inches and height h inches
21. The total cost of producing x units for which the fixed costs are \$1200 and the cost per unit is \$25
22. The total revenue obtained by selling x units at \$3.59 per unit

In Exercises 23–26, translate the statement into an algebraic expression or equation.

23. Thirty percent of the list price L
24. The amount of water in q quarts of a liquid that is 35% water
25. The percent of 500 that is represented by the number N
26. The percent change in sales from one month to the next if the monthly sales are S_1 and S_2, respectively

In Exercises 27 and 28, write an expression for the area of the region in the figure.

27. 

28. 

Number Problems **In Exercises 29–34, write a mathematical model for the number problem and solve.**

29. The sum of two consecutive natural numbers is 525. Find the numbers.
30. The sum of three consecutive natural numbers is 804. Find the numbers.
31. One positive number is 5 times another number. The difference between the two numbers is 148. Find the numbers.
32. One positive number is $\frac{1}{5}$ of another number. The difference between the two numbers is 76. Find the numbers.
33. Find two consecutive integers whose product is 5 less than the square of the smaller number.
34. Find two consecutive natural numbers such that the difference of their reciprocals is $\frac{1}{4}$ the reciprocal of the smaller number.

In Exercises 35–40, solve the percent equation.

35. What is 30% of 45?
36. What is 175% of 360?
37. 432 is what percent of 1600?
38. 459 is what percent of 340?
39. 12 is $\frac{1}{2}$% of what number?
40. 70 is 40% of what number?

41. ***Finance*** A family has annual loan payments equaling 58.6% of their annual income. During the year, their loan payments total \$13,077.75. What is their annual income?

42. ***Finance*** A salesperson's weekly paycheck is 15% less than her coworker's paycheck. The two paychecks total \$645. Find the amount of each paycheck.

43. ***Discount*** The price of a swimming pool has been discounted 16.5%. The sale price is \$1210.75. Find the original list price of the pool.

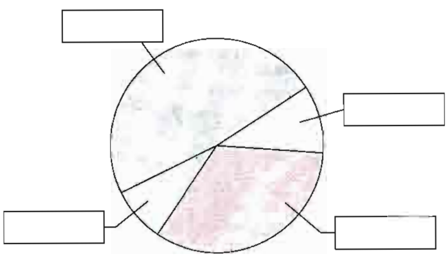
▶ Model It

44. ***Government*** The tables show the sources of income (in billions of dollars) and expenses (in billions of dollars) for the federal government in 1999. (Source: United States Office of Management and Budget)

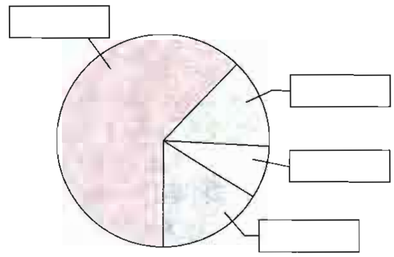
Source of income	Income
Corporation taxes	184.68
Income tax	879.48
Social Security	611.83
Other	151.46

Source of expenses	Expense
Interest on debt	229.74
Health and human services	1058.89
Defense department	274.87
Other	139.54

(a) Find the percent of the total income for each category. Then use these percents to label the circle graph. To print an enlarged copy of the graph, refer to *www.mathgraphs.com*.



(b) Find the percent of the total expenses for each category. Then use these percents to label the circle graph. To print an enlarged copy of the graph, refer to *www.mathgraphs.com*.



(c) Compare the total income and total expenses. How much of a surplus or deficit is there?

In Exercises 45–48, the values or prices of various items are given for 1980 and 2000. Find the percent change for each item. (Source: 2001 Statistical Abstract of the U.S.)

	Item	*1980*	*2000*
45.	Gallon of diesel fuel	\$0.82	\$0.94
46.	Cable TV monthly basic rate	\$7.69	\$30.08
47.	An ounce of gold	\$613.00	\$280.00
48.	New one-family home	\$64,600	\$169,000

49. ***Dimensions of a Room*** A room is 1.5 times as long as it is wide, and its perimeter is 25 meters.

(a) Draw a diagram that represents the problem. Identify the length as l and the width as w.

(b) Write l in terms of w and write an equation for the perimeter in terms of w.

(c) Find the dimensions of the room.

50. ***Dimensions of a Picture Frame*** A picture frame has a total perimeter of 2 meters. The height of the frame is 0.62 times its width.

(a) Draw a diagram that represents the problem. Identify the width as w and the height as h.

(b) Write h in terms of w and write an equation for the perimeter in terms of w.

(c) Find the dimensions of the picture frame.

51. ***Course Grade*** To get an A in a course, you must have an average of at least 90 on four tests of 100 points each. The scores on your first three tests were 87, 92, and 84. What must you score on the fourth test to get an A for the course?

52. ***Course Grade*** You are taking a course that has four tests. The first three tests are 100 points each and the fourth test is 200 points. To get an A in the course, you must have an average of at least 90% on the four tests. Your scores on the first three tests were 87, 92, and 84. What must you score on the fourth test to get an A for the course?

53. ***Travel Time*** You are driving on a Canadian freeway to a town that is 300 kilometers from your home. After 30 minutes you pass a freeway exit that you know is 50 kilometers from your home. Assuming that you continue at the same constant speed, how long will it take for the entire trip?

54. ***Travel Time*** Two cars start at an interstate interchange and travel in the same direction at average speeds of 40 miles per hour and 55 miles per hour. How much time must elapse before the two cars are 5 miles apart?

55. ***Travel Time*** On the first part of a 317-mile trip, a salesperson averaged 58 miles per hour. He averaged only 52 miles per hour on the last part of the trip because of an increased volume of traffic. The total time of the trip was 5 hours and 45 minutes. Find the amount of time at each of the two speeds.

56. ***Travel Time*** Students are traveling in two cars to a football game 135 miles away. The first car leaves on time and travels at an average speed of 45 miles per hour. The second car starts $\frac{1}{2}$ hour later and travels at an average speed of 55 miles per hour. How long will it take the second car to catch up to the first car? Will the second car catch up to the first car before the first car arrives at the game?

57. ***Travel Time*** Two families meet at a park for a picnic. At the end of the day one family travels east at an average speed of 42 miles per hour and the other travels west at an average speed of 50 miles per hour. Both families have approximately 160 miles to travel.
   (a) Find the time it takes each family to get home.
   (b) Find the time that will have elapsed when they are 100 miles apart.
   (c) Find the distance the eastbound family has to travel after the westbound family has arrived home.

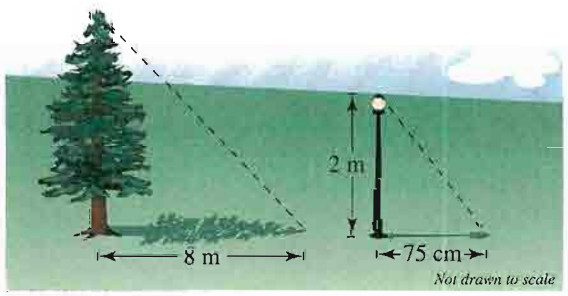
58. ***Average Speed*** A truck driver traveled at an average speed of 55 miles per hour on a 200-mile trip to pick up a load of freight. On the return trip (with the truck fully loaded), the average speed was 40 miles per hour. What was the average speed for the round trip?

59. ***Wind Speed*** An executive flew in the corporate jet to a meeting in a city 1500 kilometers away. After traveling the same amount of time on the return flight, the pilot mentioned that they still had 300 kilometers to go. The air speed of the plane was 600 kilometers per hour. How fast was the wind blowing? (Assume that the wind direction was parallel to the flight path and constant all day.)

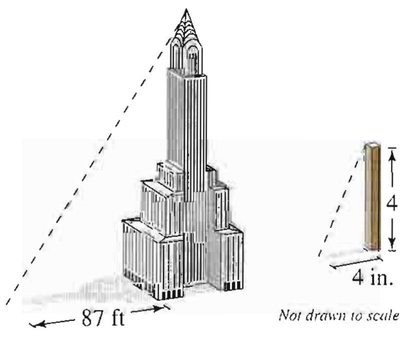
60. ***Physics*** Light travels at the speed of 3.0×10^8 meters per second. Find the time in minutes required for light to travel from the sun to Earth (a distance of 1.5×10^{11} meters).

61. ***Radio Waves*** Radio waves travel at the same speed as light, 3.0×10^8 meters per second. Find the time required for a radio wave to travel from Mission Control in Houston to NASA astronauts on the surface of the moon 3.86×10^8 meters away.

62. ***Height of a Tree*** To obtain the height of a tree (see figure), you measure the tree's shadow and find that it is 8 meters long. You also measure the shadow of a two-meter lamppost and find that it is 75 centimeters long. How tall is the tree?



63. ***Height of a Building*** To obtain the height of the Chrysler Building in New York (see figure), you measure the building's shadow and find that it is 87 feet long. You also measure the shadow of a four-foot stake and find that it is 4 inches long. How tall is the Chrysler building?



64. ***Flagpole Height*** A person who is 6 feet tall walks away from a flagpole toward the tip of the shadow of the flagpole. When the person is 30 feet from the flagpole, the tips of the person's shadow and the shadow cast by the flagpole coincide at a point 5 feet in front of the person. Find the height of the flagpole.

65. ***Shadow Length*** A person who is 6 feet tall walks away from a 50-foot silo toward the tip of the silo's shadow. At a distance of 32 feet from the silo, the person's shadow begins to emerge beyond the silo's shadow. How much farther must the person walk to be completely out of the silo's shadow?

66. ***Investment*** You plan to invest \$12,000 in two funds paying $4\frac{1}{2}\%$ and 5% simple interest. (There is more risk in the 5% fund.) Your goal is to obtain a total annual interest income of \$580 from the investments. What is the smallest amount you can invest in the 5% fund and still meet your objective?

67. ***Investment*** You plan to invest \$25,000 in two funds paying 3% and $4\frac{1}{2}\%$ simple interest. (There is more risk in the $4\frac{1}{2}\%$ fund.) Your goal is to obtain a total annual interest income of \$1000 from the investments. What is the smallest amount you can invest in the $4\frac{1}{2}\%$ fund and still to meet your objective?

68. ***Business*** A nursery has \$20,000 of inventory in dogwood trees and red maple trees. The profit on a dogwood tree is 25% and the profit on a red maple is 17%. The profit for the entire stock is 20%. How much was invested in each type of tree?

69. ***Business*** An automobile dealer has \$600,000 of inventory in compact cars and midsize cars. The profit on a compact car is 24% and the profit on a midsize car is 28%. The profit for the entire stock is 25%. How much was invested in each type of car?

70. ***Mixture Problem*** Using the values in the table, determine the amounts of solutions 1 and 2, respectively, needed to obtain the desired amount and concentration of the final mixture.

	Concentration			Amount of final solution
	Solution 1	Solution 2	Final solution	
(a)	10%	30%	25%	100 gal
(b)	25%	50%	30%	5 L
(c)	15%	45%	30%	10 qt
(d)	70%	90%	75%	25 gal

71. ***Mixture Problem*** A 100% concentrate is to be mixed with a mixture having a concentration of 40% to obtain 55 gallons of a mixture with a concentration of 75%. How much of the 100% concentrate will be needed?

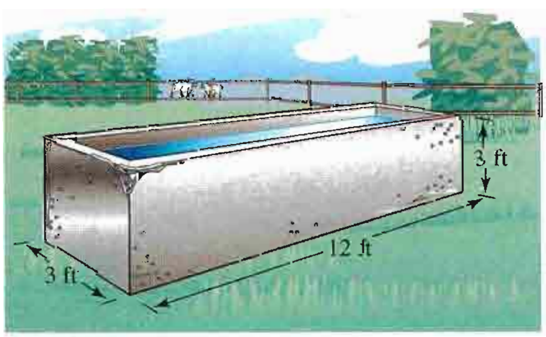
72. ***Mixture Problem*** A forester mixes gasoline and oil to make 2 gallons of mixture for his two-cycle chain-saw engine. This mixture is 32 parts gasoline and 1 part two-cycle oil. How much gasoline must be added to bring the mixture to 40 parts gasoline and 1 part oil?

73. ***Mixture Problem*** A grocer mixes peanuts that cost \$2.49 per pound and walnuts that cost \$3.89 per pound to make 100 pounds of a mixture that costs \$3.19 per pound. How much of each kind of nut is put into the mixture?

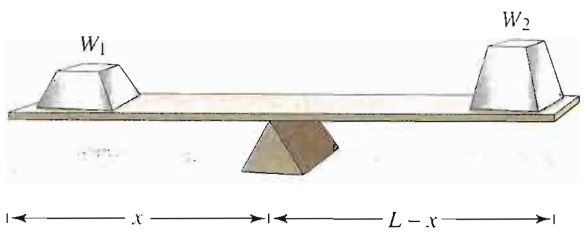
74. ***Company Costs*** An outdoor furniture manufacturer has fixed costs of \$10,000 per month and average variable costs of \$8.50 per unit manufactured. The company has \$85,000 available to cover the monthly costs. How many units can the company manufacture? (*Fixed costs* are those that occur regardless of the level of production. *Variable costs* depend on the level of production.)

75. ***Company Costs*** A plumbing supply company has fixed costs of \$10,000 per month and average variable costs of \$9.30 per unit manufactured. The company has \$85,000 available to cover the monthly costs. How many units can the company manufacture? (*Fixed costs* are those that occur regardless of the level of production. *Variable costs* depend on the level of production.)

76. ***Water Depth*** A trough is 12 feet long, 3 feet deep, and 3 feet wide (see figure). Find the depth of the water when the trough contains 70 gallons (1 gallon $\approx$ 0.13368 cubic foot).



Physics **In Exercises 77 and 78, you have a uniform beam of length L with a fulcrum x feet from one end (see figure). Objects with weights W_1 and W_2 are placed at opposite ends of the beam. The beam will balance when $W_1x = W_2(L - x)$. Find x such that the beam will balance.**



77. Two children weighing 50 pounds and 75 pounds are playing on a seesaw that is 10 feet long.

78. A person weighing 200 pounds is attempting to move a 550-pound rock with a bar that is 5 feet long.

In Exercises 79–90, solve for the indicated variable.

79. *Area of a Triangle*
Solve for h: $A = \frac{1}{2}bh$

80. *Volume of a Right Circular Cylinder*
Solve for h: $V = \pi r^2 h$

81. *Markup* Solve for C: $S = C + RC$

82. *Investment at Simple Interest*
Solve for r: $A = P + Prt$

83. *Volume of an Oblate Spheroid*
Solve for b: $V = \frac{4}{3}\pi a^2 b$

84. *Volume of a Spherical Segment*
Solve for r: $V = \frac{1}{3}\pi h^2(3r - h)$

85. *Free-Falling Body* Solve for a: $h = v_0 t + \frac{1}{2}at^2$

86. *Lensmaker's Equation*
Solve for R_1: $\frac{1}{f} = (n - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$

87. *Capacitance in Series Circuits*
Solve for C_1: $C = \dfrac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$

88. *Arithmetic Progression*
Solve for a: $S = \frac{n}{2}[2a + (n - 1)d]$

89. *Arithmetic Progression*
Solve for n: $L = a + (n - 1)d$

90. *Geometric Progression* Solve for r: $S = \dfrac{rL - a}{r - 1}$

91. *Volume of a Billiard Ball* A billiard ball has a volume of 5.96 cubic inches. Find the radius of a billiard ball.

92. *Length of a Tank* The diameter of a cylindrical propane gas tank is 4 feet. The total volume of the tank is 603.2 cubic feet. Find the length of the tank.

Synthesis

True or False? **In Exercises 93 and 94, determine whether the statement is true or false. Justify your answer.**

93. "8 less than z cubed divided by the difference of z squared and 9" can be written as

$$\frac{z^3 - 8}{(z - 9)^2}.$$

94. The volume of a cube with a side of length 9.5 inches is greater than the volume of a sphere with a radius of 5.9 inches.

95. Consider the linear equation $ax + b = 0$.
(a) What is the sign of the solution if $ab > 0$?
(b) What is the sign of the solution if $ab < 0$?
In each case, explain your reasoning.

96. Write a linear equation that has the solution $x = -3$. (There are many correct answers.)

Review

In Exercises 97–100, simplify the expression.

97. $(5x^4)(25x^2)^{-1}$, $x \neq 0$

98. $\sqrt{150s^2t^3}$

99. $\dfrac{3}{x - 5} + \dfrac{2}{5 - x}$

100. $\dfrac{5}{x} + \dfrac{3x}{x^2 - 9} - \dfrac{10}{x + 3}$

In Exercises 101–104, rationalize the denominator.

101. $\dfrac{10}{7\sqrt{3}}$

102. $\dfrac{6}{\sqrt{10} - 2}$

103. $\dfrac{5}{\sqrt{6} + \sqrt{11}}$

104. $\dfrac{14}{3\sqrt{10} - 1}$

1.4 Quadratic Equations

▶ What you should learn

- How to solve quadratic equations by factoring
- How to solve quadratic equations by extracting square roots
- How to solve quadratic equations by completing the square
- How to use the Quadratic Formula to solve quadratic equations
- How to use quadratic equations to model and solve real-life problems

▶ Why you should learn it

Quadratic equations can be used to model and solve real-life problems. For instance, in Exercise 118 on page 120, you will use a quadratic equation to model the time it takes an object to fall from the top of the CN Tower.

Chris Thomaidis/Getty Images

Factoring

A **quadratic equation** in x is an equation that can be written in the general form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers with $a \neq 0$. A quadratic equation in x is also known as a **second-degree polynomial equation** in x.

In this section, you will study four methods for solving quadratic equations: *factoring*, *extracting square roots*, *completing the square*, and the *Quadratic Formula*. The first method is based on the Zero-Factor Property from Section P.1.

If $ab = 0$, then $a = 0$ or $b = 0$. Zero-Factor Property

To use this property, write the left side of the general form of a quadratic equation as the product of two linear factors. Then find the solutions of the quadratic equation by setting each linear factor equal to zero.

Example 1 ▶ Solving a Quadratic Equation by Factoring

a. $2x^2 + 9x + 7 = 3$ Original equation

$2x^2 + 9x + 4 = 0$ Write in general form.

$(2x + 1)(x + 4) = 0$ Factor.

$2x + 1 = 0 \Rightarrow x = -\dfrac{1}{2}$ Set 1st factor equal to 0.

$x + 4 = 0 \Rightarrow x = -4$ Set 2nd factor equal to 0.

The solutions are $x = -\frac{1}{2}$ and $x = -4$. Check these in the original equation.

b. $6x^2 - 3x = 0$ Original equation

$3x(2x - 1) = 0$ Factor.

$3x = 0 \Rightarrow x = 0$ Set 1st factor equal to 0.

$2x - 1 = 0 \Rightarrow x = \dfrac{1}{2}$ Set 2nd factor equal to 0.

The solutions are $x = 0$ and $x = \frac{1}{2}$. Check these in the original equation.

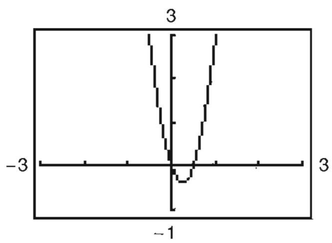
Be sure you see that the Zero-Factor Property works *only* for equations written in general form (in which the right side of the equation is zero). So, all terms must be collected on one side *before* factoring. For instance, in the equation

$$(x - 5)(x + 2) = 8$$

it is *incorrect* to set each factor equal to 8. Try to solve this equation correctly.

Technology

You can use a graphing utility to check graphically the real solutions of a quadratic equation. Begin by writing the equation in general form. Then set y equal to the left side and graph the resulting equation. The x-intercepts of the equation represent the real solutions of the original equation. For example, to check the solutions of $6x^2 - 3x = 0$, graph $y = 6x^2 - 3x$, as shown below. Notice that the x-intercepts occur at $x = 0$ and $x = \frac{1}{2}$, as found in Example 1(b).



Extracting Square Roots

There is a nice shortcut for solving quadratic equations of the form $u^2 = d$, where $d > 0$ and u is an algebraic expression. By factoring, you can see that this equation has two solutions.

$$u^2 = d \quad \text{Write original equation.}$$

$$u^2 - d = 0 \quad \text{Write in general form.}$$

$$(u + \sqrt{d})(u - \sqrt{d}) = 0 \quad \text{Factor.}$$

$$u + \sqrt{d} = 0 \Rightarrow u = -\sqrt{d} \quad \text{Set 1st factor equal to 0.}$$

$$u - \sqrt{d} = 0 \Rightarrow u = \sqrt{d} \quad \text{Set 2nd factor equal to 0.}$$

Because the two solutions differ only in sign, you can write the solutions together, using a "plus or minus sign," as

$$u = \pm\sqrt{d}.$$

This form of the solution is read as "u is equal to plus or minus the square root of d." Solving an equation of the form $u^2 = d$ without going through the steps of factoring is called **extracting square roots.**

Extracting Square Roots

The equation $u^2 = d$, where $d > 0$, has exactly two solutions:

$$u = \sqrt{d} \quad \text{and} \quad u = -\sqrt{d}.$$

These solutions can also be written as

$$u = \pm\sqrt{d}.$$

Example 2 ▶ Extracting Square Roots

Solve each equation by extracting square roots.

a. $4x^2 = 12$ **b.** $(x - 3)^2 = 7$

Solution

a. $4x^2 = 12$ Write original equation.

$x^2 = 3$ Divide each side by 4.

$x = \pm\sqrt{3}$ Extract square roots.

The solutions are $x = \sqrt{3}$ and $x = -\sqrt{3}$. Check these in the original equation.

b. $(x - 3)^2 = 7$ Write original equation.

$x - 3 = \pm\sqrt{7}$ Extract square roots.

$x = 3 \pm \sqrt{7}$ Add 3 to each side.

The solutions are $x = 3 \pm \sqrt{7}$. Check these in the original equation.

Completing the Square

The equation in Example 2(b) was given in the form $(x - 3)^2 = 7$ so that you could find the solution by extracting square roots. Suppose, however, that the equation had been given in the general form $x^2 - 6x + 2 = 0$. Because this equation is equivalent to the original, it has the same two solutions, $x = 3 \pm \sqrt{7}$. However, the left side of the equation is not factorable, and you cannot find its solutions unless you rewrite the equation by **completing the square.**

Completing the Square

To **complete the square** for the expression $x^2 + bx$, add $(b/2)^2$, which is the square of half the coefficient of x. Consequently,

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2.$$

When solving quadratic equations by completing the square, you must add $(b/2)^2$ to *each side* in order to maintain equality. If the leading coefficient is *not* 1, you must divide each side of the equation by the leading coefficient *before* completing the square, as shown in Example 3.

Example 3 ▶ Completing the Square

Solve $3x^2 - 4x - 5 = 0$ by completing the square.

Solution

$$3x^2 - 4x - 5 = 0 \qquad \text{Write original equation.}$$

$$3x^2 - 4x = 5 \qquad \text{Add 5 to each side.}$$

$$x^2 - \frac{4}{3}x = \frac{5}{3} \qquad \text{Divide each side by 3.}$$

$$x^2 - \frac{4}{3}x + \left(-\frac{2}{3}\right)^2 = \frac{5}{3} + \left(-\frac{2}{3}\right)^2 \qquad \text{Add } \left(-\tfrac{2}{3}\right)^2 \text{ to each side.}$$

$\left(\text{half of } -\frac{4}{3}\right)^2$

$$x^2 - \frac{4}{3}x + \frac{4}{9} = \frac{19}{9} \qquad \text{Simplify.}$$

$$\left(x - \frac{2}{3}\right)^2 = \frac{19}{9} \qquad \text{Perfect square trinomial.}$$

$$x - \frac{2}{3} = \pm\frac{\sqrt{19}}{3} \qquad \text{Extract square roots.}$$

$$x = \frac{2}{3} \pm \frac{\sqrt{19}}{3} \qquad \text{Add } \tfrac{2}{3} \text{ to each side.}$$

The solutions are $x = \dfrac{2 \pm \sqrt{19}}{3}$. Check these in the original equation.

The Quadratic Formula

Often in mathematics you are taught the long way of solving a problem first. Then, the longer method is used to develop shorter techniques. The long way stresses understanding and the short way stresses efficiency.

For instance, you can think of completing the square as a "long way" of solving a quadratic equation. When you use completing the square to solve a quadratic equation, you must complete the square for *each* equation separately. In the following derivation, you complete the square *once* in a general setting to obtain the **Quadratic Formula**—a shortcut for solving a quadratic equation.

$$ax^2 + bx + c = 0 \qquad \text{Write in general form, } a \neq 0.$$

$$ax^2 + bx = -c \qquad \text{Subtract } c \text{ from each side.}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{Divide each side by } a.$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \text{Complete the square.}$$

$\left(\text{half of } \frac{b}{a}\right)^2$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \text{Simplify.}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{Extract square roots.}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2|a|} \qquad \text{Solutions}$$

Note that because $\pm 2|a|$ represents the same numbers as $\pm 2a$, you can omit the absolute value sign. So, the formula simplifies to

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

STUDY TIP

You can solve every quadratic equation by completing the square or using the Quadratic Formula.

The Quadratic Formula

The solutions of a quadratic equation in the general form

$$ax^2 + bx + c = 0, \qquad a \neq 0$$

are given by the **Quadratic Formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

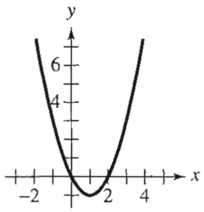
The Quadratic Formula is one of the most important formulas in algebra. You should learn the verbal statement of the Quadratic Formula:

"Negative b, plus or minus the square root of b squared minus $4ac$, *all* divided by $2a$."

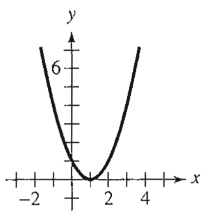
Exploration

From each graph, can you tell whether the discriminant is positive, zero, or negative? Explain your reasoning. Find each discriminant to verify your answers.

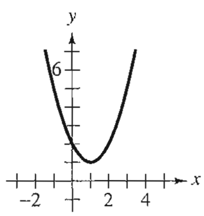
a. $x^2 - 2x = 0$



b. $x^2 - 2x + 1 = 0$



c. $x^2 - 2x + 2 = 0$



How many solutions would part (c) have if the linear term was $2x$? If the constant was -2?

In the Quadratic Formula, the quantity under the radical sign, $b^2 - 4ac$, is called the **discriminant** of the quadratic expression $ax^2 + bx + c$. It can be used to determine the nature of the solutions of a quadratic equation.

Solutions of a Quadratic Equation

The solutions of a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, can be classified as follows. If the discriminant $b^2 - 4ac$ is

1. *positive*, then the quadratic equation has *two* distinct real solutions and its graph has *two* x-intercepts.
2. *zero*, then the quadratic equation has *one* repeated real solution and its graph has *one* x-intercept.
3. *negative*, then the quadratic equation has *no* real solutions and its graph has *no* x-intercepts.

If the discriminant of a quadratic equation is negative, as in case 3 above, then its square root is imaginary (not a real number) and the Quadratic Formula yields two complex solutions. You will study complex solutions in Section 1.5.

When using the Quadratic Formula, remember that *before* the formula can be applied, you must first write the quadratic equation in general form.

Example 4 ▶ The Quadratic Formula: Two Distinct Solutions

Use the Quadratic Formula to solve $x^2 + 3x = 9$.

Solution

The general form of the equation is $x^2 + 3x - 9 = 0$. The discriminant is $b^2 - 4ac = 9 + 36 = 45$, which is positive. So, the equation has two real solutions. You can solve the equation as follows.

$$x^2 + 3x - 9 = 0 \qquad \text{General form}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic Formula}$$

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-9)}}{2(1)} \qquad \text{Substitute } a = 1, b = 3, \text{ and } c = -9.$$

$$x = \frac{-3 \pm \sqrt{45}}{2} \qquad \text{Simplify.}$$

$$x = \frac{-3 \pm 3\sqrt{5}}{2} \qquad \text{Simplify.}$$

The two solutions are:

$$x = \frac{-3 + 3\sqrt{5}}{2} \quad \text{and} \quad x = \frac{-3 - 3\sqrt{5}}{2}.$$

Check these in the original equation.

Applications

Quadratic equations often occur in problems dealing with area. Here is a simple example. "A square room has an area of 144 square feet. Find the dimensions of the room." To solve this problem, let x represent the length of each side of the room. Then, by solving the equation

$$x^2 = 144$$

you can conclude that each side of the room is 12 feet long. Note that although the equation $x^2 = 144$ has two solutions, $x = -12$ and $x = 12$, the negative solution does not make sense in the context of the problem, so you choose the positive solution.

Example 5 ▶ Finding the Dimensions of a Room

A bedroom is 3 feet longer than it is wide (see Figure 1.19) and has an area of 154 square feet. Find the dimensions of the room.

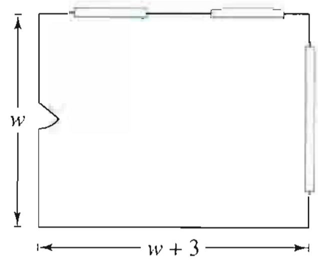


FIGURE 1.19

Solution

Verbal Model: $\text{Width of room} \cdot \text{Length of room} = \text{Area of room}$

Labels:
Width of room $= w$ (feet)
Length of room $= w + 3$ (feet)
Area of room $= 154$ (square feet)

Equation:

$$w(w + 3) = 154$$

$$w^2 + 3w - 154 = 0$$

$$(w - 11)(w + 14) = 0$$

$$w - 11 = 0 \quad \Rightarrow \quad w = 11$$

$$w + 14 = 0 \quad \Rightarrow \quad w = -14$$

Choosing the positive value, you find that the width is 11 feet and the length is $w + 3$, or 14 feet. You can check this solution by observing that the length is 3 feet longer than the width *and* that the product of the length and width is 154 square feet.

Another common application of quadratic equations involves an object that is falling (or projected into the air). The general equation that gives the height of such an object is called a **position equation,** and on the *Earth's* surface it has the form

$$s = -16t^2 + v_0t + s_0.$$

In this equation, s represents the height of the object (in feet), v_0 represents the initial velocity of the object (in feet per second), s_0 represents the initial height of the object (in feet), and t represents the time (in seconds).

Example 6 ▶ Falling Time

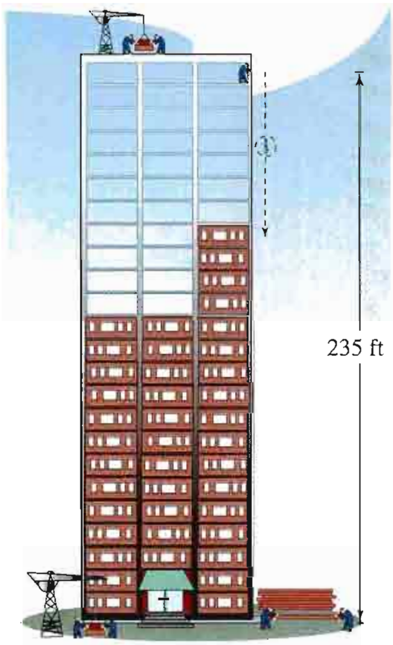


FIGURE 1.20

A construction worker on the 24th floor of a building (see Figure 1.20) accidentally drops a wrench and yells "Look out below!" Could a person at ground level hear this warning in time to get out of the way?

Solution

Assume that each floor of the building is 10 feet high, so that the wrench is dropped from a height of 235 feet (the construction worker's hand is 5 feet below the ceiling of the 24th floor). Because sound travels at about 1100 feet per second, it follows that a person at ground level hears the warning within 1 second of the time the wrench is dropped. To set up a mathematical model for the height of the wrench, use the position equation

$$s = -16t^2 + v_0t + s_0.$$

Because the object is dropped rather than thrown, the initial velocity is $v_0 = 0$ feet per second. Moreover, because the initial height is $s_0 = 235$ feet, you have the following model.

$$s = -16t^2 + (0)t + 235 = -16t^2 + 235$$

After falling for 1 second, the height of the wrench is $-16(1)^2 + 235 = 219$ feet. After falling for 2 seconds, the height of the wrench is $-16(2)^2 + 235 = 171$ feet. To find the number of seconds it takes the wrench to hit the ground, let the height s be zero and solve the equation for t.

$$s = -16t^2 + 235 \qquad \text{Write position equation.}$$

$$0 = -16t^2 + 235 \qquad \text{Substitute 0 for height.}$$

$$16t^2 = 235 \qquad \text{Add } 16t^2 \text{ to each side.}$$

$$t^2 = \frac{235}{16} \qquad \text{Divide each side by 16.}$$

$$t = \frac{\sqrt{235}}{4} \qquad \text{Extract positive square root.}$$

$$t \approx 3.83 \qquad \text{Use a calculator.}$$

The wrench will take about 3.83 seconds to hit the ground. If the person hears the warning 1 second after the wrench is dropped, the person still has almost 3 seconds to get out of the way.

STUDY TIP

The position equation used in Example 6 ignores air resistance. This implies that it is appropriate to use the position equation only to model falling objects that have little air resistance and that fall over short distances.

Example 7 ▶ Quadratic Modeling: Number of Lawyers

From 1983 to 1999, the number of lawyers L (in thousands) in the United States can be modeled by the quadratic equation

$$L = 0.008t^2 + 19.59t + 552.8, \quad 3 \le t \le 19$$

where t is the time in years, with $t = 3$ corresponding to 1983. The number of lawyers is shown graphically in Figure 1.21. According to this model, in which year will the number of lawyers reach or surpass 1 million? (Source: U.S. Bureau of Labor Statistics)

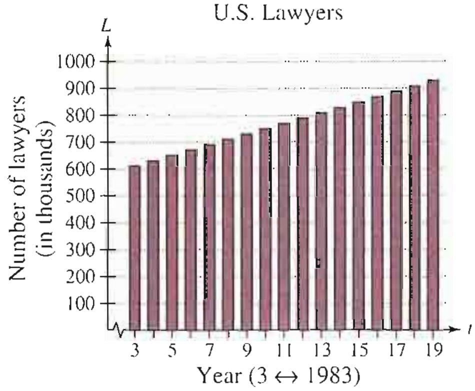


FIGURE 1.21

Solution

To find the year in which the number of lawyers will reach 1 million, you need to solve the equation

$$0.008t^2 + 19.59t + 552.8 = 1000.$$

To begin, write the equation in general form.

$$0.008t^2 + 19.59t - 447.2 = 0$$

Then apply the Quadratic Formula.

$$t = \frac{-19.59 \pm \sqrt{(19.59)^2 - 4(0.008)(-447.2)}}{2(0.008)}$$

Choosing the positive solution, you find that

$$t = \frac{-19.59 + \sqrt{(19.59)^2 - 4(0.008)(-447.2)}}{2(0.008)}$$

$$= \frac{-19.59 + \sqrt{398.08}}{0.016}$$

$$\approx 22.62.$$

Because $t = 3$ corresponds to 1983, it follows that $t \approx 22.62$ must correspond to 2002. So, the number of lawyers should have reached 1 million during the year 2002.

A fourth type of application that often involves a quadratic equation is one dealing with the hypotenuse of a right triangle. These types of applications often use the Pythagorean Theorem, which states that

$$a^2 + b^2 = c^2 \qquad \text{Pythagorean Theorem}$$

where a and b are the legs of a right triangle and c is the hypotenuse.

Example 8 ▶ An Application Involving the Pythagorean Theorem

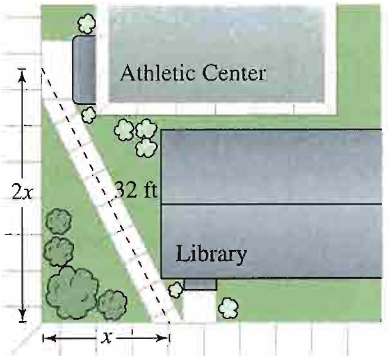


FIGURE 1.22

An L-shaped sidewalk from the athletic center to the library on a college campus is shown in Figure 1.22. The sidewalk was constructed so that the length of one sidewalk forming the L is twice as long as the other. The length of the diagonal sidewalk that cuts across the grounds between the two buildings is 32 feet. How many feet does a person save by walking on the diagonal sidewalk?

Solution

Using the Pythagorean Theorem, you have

$$x^2 + (2x)^2 = 32^2 \qquad \text{Pythagorean Theorem}$$

$$5x^2 = 1024 \qquad \text{Combine like terms.}$$

$$x^2 = 204.8 \qquad \text{Divide each side by 5.}$$

$$x = \sqrt{204.8} \qquad \text{Extract positive square root.}$$

The total distance covered by walking on the L-shaped sidewalk is

$$x + 2x = 3x$$

$$= 3\sqrt{204.8}$$

$$\approx 42.9 \text{ feet.}$$

Walking on the diagonal sidewalk saves a person about $42.9 - 32 = 10.9$ feet.

Writing ABOUT MATHEMATICS

Comparing Solution Methods In this section, you studied four algebraic methods for solving quadratic equations. Solve each of the quadratic equations below in several different ways. Write a short paragraph explaining which method(s) you prefer. Does your preferred method depend on the equation?

a. $x^2 - 4x - 5 = 0$

b. $x^2 - 4x = 0$

c. $x^2 - 4x - 3 = 0$

d. $x^2 - 4x - 6 = 0$

1.4 Exercises

In Exercises 1–6, write the quadratic equation in general form.

1. $2x^2 = 3 - 8x$
2. $x^2 = 16x$
3. $(x - 3)^2 = 3$
4. $13 - 3(x + 7)^2 = 0$
5. $\frac{1}{5}(3x^2 - 10) = 18x$
6. $x(x + 2) = 5x^2 + 1$

In Exercises 7–20, solve the quadratic equation by factoring.

7. $6x^2 + 3x = 0$
8. $9x^2 - 1 = 0$
9. $x^2 - 2x - 8 = 0$
10. $x^2 - 10x + 9 = 0$
11. $x^2 + 10x + 25 = 0$
12. $4x^2 + 12x + 9 = 0$
13. $3 + 5x - 2x^2 = 0$
14. $2x^2 = 19x + 33$
15. $x^2 + 4x = 12$
16. $-x^2 + 8x = 12$
17. $\frac{3}{4}x^2 + 8x + 20 = 0$
18. $\frac{1}{8}x^2 - x - 16 = 0$
19. $x^2 + 2ax + a^2 = 0$
20. $(x + a)^2 - b^2 = 0$

In Exercises 21–34, solve the equation by extracting square roots. List both the exact solution and the decimal solution rounded to two decimal places.

21. $x^2 = 49$
22. $x^2 = 169$
23. $x^2 = 11$
24. $x^2 = 32$
25. $3x^2 = 81$
26. $9x^2 = 36$
27. $(x - 12)^2 = 16$
28. $(x + 13)^2 = 25$
29. $(x + 2)^2 = 14$
30. $(x - 5)^2 = 30$
31. $(2x - 1)^2 = 18$
32. $(4x + 7)^2 = 44$
33. $(x - 7)^2 = (x + 3)^2$
34. $(x + 5)^2 = (x + 4)^2$

In Exercises 35–44, solve the quadratic equation by completing the square.

35. $x^2 - 2x = 0$
36. $x^2 + 4x = 0$
37. $x^2 + 4x - 32 = 0$
38. $x^2 - 2x - 3 = 0$
39. $x^2 + 6x + 2 = 0$
40. $x^2 + 8x + 14 = 0$
41. $9x^2 - 18x = -3$
42. $9x^2 - 12x = 14$
43. $8 + 4x - x^2 = 0$
44. $4x^2 - 4x - 99 = 0$

In Exercises 45–50, rewrite the quadratic portion of the algebraic expression as the sum or difference of two squares by completing the square.

45. $\dfrac{1}{x^2 + 2x + 5}$
46. $\dfrac{1}{x^2 - 12x + 19}$
47. $\dfrac{4}{x^2 + 4x - 3}$
48. $\dfrac{5}{x^2 + 25x + 11}$
49. $\dfrac{1}{\sqrt{6x - x^2}}$
50. $\dfrac{1}{\sqrt{16 - 6x - x^2}}$

Graphical Analysis In Exercises 51–58, use a graphing utility to graph the equation. Use the graph to approximate any x-intercepts of the graph. Set $y = 0$ and solve the resulting equation. Compare the result with the x-intercepts of the graph.

51. $y = (x + 3)^2 - 4$
52. $y = (x - 4)^2 - 1$
53. $y = 1 - (x - 2)^2$
54. $y = 9 - (x - 8)^2$
55. $y = -4x^2 + 4x + 3$
56. $y = 4x^2 - 1$
57. $y = x^2 + 3x - 4$
58. $y = x^2 - 5x - 24$

In Exercises 59–66, use the discriminant to determine the number of real solutions of the quadratic equation.

59. $2x^2 - 5x + 5 = 0$
60. $-5x^2 - 4x + 1 = 0$
61. $2x^2 - x - 1 = 0$
62. $x^2 - 4x + 4 = 0$
63. $\frac{1}{3}x^2 - 5x + 25 = 0$
64. $\frac{4}{7}x^2 - 8x + 28 = 0$
65. $0.2x^2 + 1.2x - 8 = 0$
66. $9 + 2.4x - 8.3x^2 = 0$

In Exercises 67–90, use the Quadratic Formula to solve the equation.

67. $2x^2 + x - 1 = 0$
68. $2x^2 - x - 1 = 0$
69. $16x^2 + 8x - 3 = 0$
70. $25x^2 - 20x + 3 = 0$
71. $2 + 2x - x^2 = 0$
72. $x^2 - 10x + 22 = 0$
73. $x^2 + 14x + 44 = 0$
74. $6x = 4 - x^2$
75. $x^2 + 8x - 4 = 0$
76. $4x^2 - 4x - 4 = 0$
77. $12x - 9x^2 = -3$
78. $16x^2 + 22 = 40x$
79. $9x^2 + 24x + 16 = 0$
80. $36x^2 + 24x - 7 = 0$
81. $4x^2 + 4x = 7$
82. $16x^2 - 40x + 5 = 0$
83. $28x - 49x^2 = 4$
84. $3x + x^2 - 1 = 0$
85. $8t = 5 + 2t^2$
86. $25h^2 + 80h + 61 = 0$

87. $(y - 5)^2 = 2y$

88. $(z + 6)^2 = -2z$

89. $\frac{1}{2}x^2 + \frac{3}{8}x = 2$

90. $(\frac{5}{7}x - 14)^2 = 8x$

In Exercises 91–98, use the Quadratic Formula to solve the equation. (Round your answer to three decimal places.)

91. $5.1x^2 - 1.7x - 3.2 = 0$

92. $2x^2 - 2.50x - 0.42 = 0$

93. $-0.067x^2 - 0.852x + 1.277 = 0$

94. $-0.005x^2 + 0.101x - 0.193 = 0$

95. $422x^2 - 506x - 347 = 0$

96. $1100x^2 + 326x - 715 = 0$

97. $12.67x^2 + 31.55x + 8.09 = 0$

98. $-3.22x^2 - 0.08x + 28.651 = 0$

In Exercises 99–108, solve the equation using any convenient method.

99. $x^2 - 2x - 1 = 0$

100. $11x^2 + 33x = 0$

101. $(x + 3)^2 = 81$

102. $x^2 - 14x + 49 = 0$

103. $x^2 - x - \frac{11}{4} = 0$

104. $x^2 + 3x - \frac{3}{4} = 0$

105. $(x + 1)^2 = x^2$

106. $a^2x^2 - b^2 = 0$

107. $3x + 4 = 2x^2 - 7$

108. $4x^2 + 2x + 4 = 2x + 8$

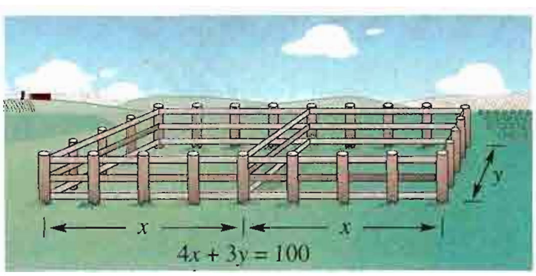
109. ***Floor Space*** The floor of a one-story building is 14 feet longer than it is wide. The building has 1632 square feet of floor space.

(a) Draw a diagram that gives a visual representation of the floor space. Represent the width as w and show the length in terms of w.

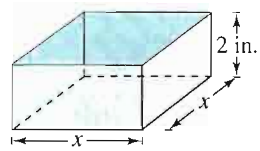
(b) Write a quadratic equation in terms of w.

(c) Find the length and width of the floor of the building.

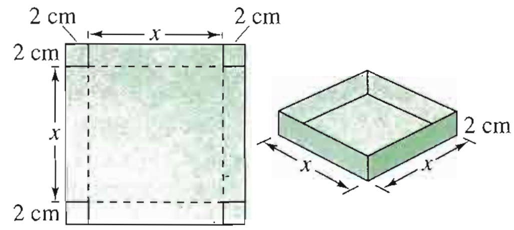
110. ***Dimensions of a Corral*** A rancher has 100 meters of fencing to enclose two adjacent rectangular corrals (see figure). The rancher wants the enclosed area to be 350 square meters. What dimensions should the rancher use to obtain this area?



111. ***Packaging*** An open box with a square base (see figure) is to be constructed from 84 square inches of material. The height of the box is 2 inches. What are the dimensions of the box? (*Hint:* The surface area is $S = x^2 + 4xh$.)



112. ***Packaging*** An open box is to be made from a square piece of material by cutting two-centimeter squares from the corners and turning up the sides (see figure). The volume of the finished box is to be 200 cubic centimeters. Find the size of the original piece of material.



113. ***Mowing the Lawn*** Two landscapers must mow a rectangular lawn that measures 100 feet by 200 feet. Each wants to mow no more than half of the lawn. The first starts by mowing around the outside of the lawn. How wide a strip must the first landscaper mow on each of the four sides in order to mow no more than half of the lawn? The mower has a 24-inch cut. Approximate the required number of trips around the lawn.

114. ***Seating*** A rectangular classroom seats 72 students. If the seats were rearranged with three more seats in each row, the classroom would have two fewer rows. Find the original number of seats in each row.

In Exercises 115–118, use the position equation given in Example 6 as the model for the problem.

115. ***Military*** A B-52 stratofortress flying at 32,000 feet over level terrain drops a 500-pound bomb.

(a) How long will it take until the bomb strikes the ground?

(b) The plane is flying at 600 miles per hour. How far will the bomb travel horizontally during its descent?

116. ***Eiffel Tower*** You drop a coin from the top of the Eiffel Tower in Paris. The building has a height of 984 feet.

(a) Use the position equation to write a mathematical model for the height of the coin.

(b) Find the height of the coin after 4 seconds.

(c) How long will it take before the coin strikes the ground?

117. ***Sports*** You throw a baseball straight up into the air at a velocity of 45 feet per second. You release the baseball at a height of 5.5 feet and catch it when it falls back to a height of 6 feet.

(a) Use the position equation to write a mathematical model for the height of the baseball.

(b) What is the height of the baseball after 0.5 second?

(c) How many seconds is the baseball in the air?

▶ Model It

118. ***CN Tower*** At 1815 feet tall, the CN Tower in Toronto, Ontario is the world's tallest self-supporting structure. An object is dropped from the top of the tower.

(a) Use the position equation to write a model for the height of the object.

(b) Complete the table.

Time, t	0	2	4	6	8	10	12
Height, s							

(c) From the table in part (b), determine the time interval during which the object reaches the ground. Numerically approximate the time it takes the object to reach the ground.

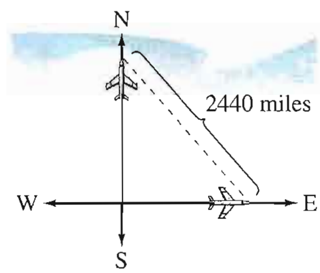
(d) Find the time it takes the object to reach the ground algebraically. How close was your numerical approximation?

(e) Use a graphing utility with the appropriate viewing window to verify your answer to parts (c) and (d).

119. ***Geometry*** The hypotenuse of an isosceles right triangle is 5 centimeters long. How long are its sides?

120. ***Geometry*** An equilateral triangle has a height of 10 inches. How long is one of its sides? (*Hint:* Use the height of the triangle to partition the triangle into two congruent right triangles.)

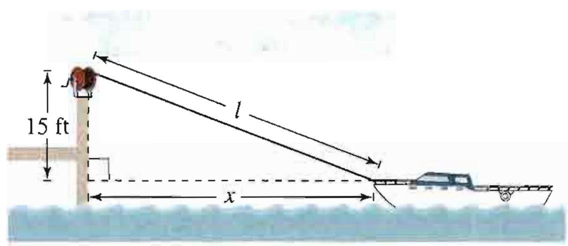
121. ***Flying Speed*** Two planes leave simultaneously from Chicago's O'Hare Airport, one flying due north and the other due east (see figure). The northbound plane is flying 50 miles per hour faster than the eastbound plane. After 3 hours, the planes are 2440 miles apart. Find the speed of each plane.



122. ***Boating*** A winch is used to tow a boat to a dock. The rope is attached to the boat at a point 15 feet below the level of the winch (see figure).

(a) Use the Pythagorean Theorem to write an equation giving the relationship between l and x.

(b) Find the distance from the boat to the dock when there is 75 feet of rope out.



123. ***Demand*** The demand equation for a product is $p = 20 - 0.0002x$, where p is the price per unit and x is the number of units sold. The total revenue for selling x units is

$$\text{Revenue} = xp = x(20 - 0.0002x).$$

How many units must be sold to produce a revenue of \$500,000?

124. ***Demand*** The demand equation for a product is $p = 60 - 0.0004x$, where p is the price per unit and x is the number of units sold. The total revenue for selling x units is

$$\text{Revenue} = xp = x(60 - 0.0004x).$$

How many units must be sold to produce a revenue of \$220,000?

Cost **In Exercises 125–128, use the cost equation to find the number of units *x* that a manufacturer can produce for the given cost *C*. Round your answer to the nearest positive integer.**

125. $C = 0.125x^2 + 20x + 500$ $\quad C = \$14{,}000$

126. $C = 0.5x^2 + 15x + 5000$ $\quad C = \$11{,}500$

127. $C = 800 + 0.04x + 0.002x^2$ $\quad C = \$1680$

128. $C = 800 - 10x + \dfrac{x^2}{4}$ $\quad C = \$896$

129. ***Population Statistics*** The population of the United States from 1800 to 1890 can be approximated by the model

$$\text{Population} = 0.68522x^2 + 0.0871x + 6.047$$

where the population is given in millions of people and t represents time, with $t = 0$ corresponding to 1800, $t = 1$ corresponding to 1810, and so on. If this model had continued to be valid up through the present time, when would the population of the United States have reached 250 million? Judging from the graph, would you say this model was a good representation of the population through 1890? (Source: U.S. Bureau of the Census)

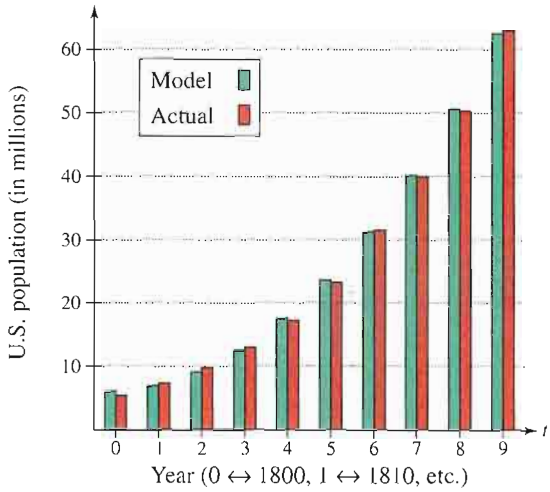


FIGURE FOR 129

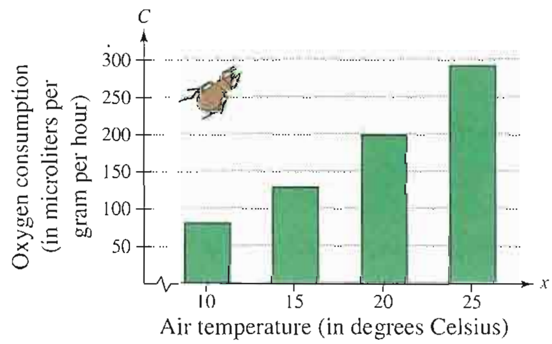
130. ***Biology*** The metabolic rate of an ectothermic organism increases with increasing temperature within a certain range. The graph shows experimental data for the oxygen consumption C (in microliters per gram per hour) of a beetle at certain temperatures. This data can be approximated by the model

$$C = 0.45x^2 - 1.65x + 50.75, \qquad 10 \le x \le 25$$

where x is the air temperature in degrees Celsius.

(a) The oxygen consumption is 150 microliters per gram per hour. What is the air temperature?

(b) The temperature is increased from 10°C to 20°C. The oxygen consumption is increased by approximately what factor?



131. ***Boating*** The total amount S (in billions of dollars) spent on pleasure boats in the United States from 1992 through 1999 can be approximated by the model

$$S = -0.0959t^2 + 1.770t + 2.29$$

where t is the time, with $t = 2$ corresponding to 1992. (Source: National Sporting Goods Association)

(a) Use a graphing utility to graph the model over the interval $2 \le t \le 9$.

(b) If the model is used to forecast future sales, will sales ever exceed 12 billion dollars? If so, estimate the year.

132. ***Flying Distance*** A commercial jet flies to three cities whose locations form the vertices of a right triangle (see figure). The total flight distance (from Oklahoma City to Austin to New Orleans and back to Oklahoma City) is approximately 1348 miles. It is 560 miles between Oklahoma City and New Orleans. Approximate the other two distances.

Synthesis

True or False? **In Exercises 133 and 134, determine whether the statement is true or false. Justify your answer.**

133. The quadratic equation $-3x^2 - x = 10$ has two real solutions.

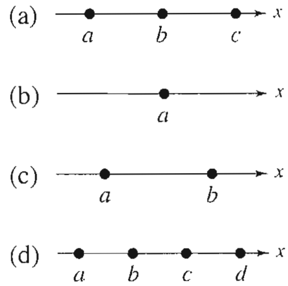
134. If $(2x - 3)(x + 5) = 8$, then either $2x - 3 = 8$ or $x + 5 = 8$.

135. To solve the equation

$$2x^2 + 3x = 15x$$

a student divides each side by x and solves the equation $2x + 3 = 15$. The resulting solution $(x = 6)$ satisfies the original equation. Is there an error? Explain.

136. The graphs show the solutions of equations plotted on the real number line. In each case, determine whether the solution(s) is (are) for a linear equation, a quadratic equation, both, or neither. Explain.



137. Solve $3(x + 4)^2 + (x + 4) - 2 = 0$ in two ways.

(a) Let $u = x + 4$, and solve the resulting equation for u. Then solve the u-solution for x.

(b) Expand and collect like terms in the equation, and solve the resulting equation for x.

(c) Which method is easier? Explain.

138. Solve the equations, given that a and b are not zero.

(a) $ax^2 + bx = 0$

(b) $ax^2 - ax = 0$

Think About It **In Exercises 139–142, write a quadratic equation that has the given solutions. (There are many correct answers.)**

139. -3 and 6

140. -4 and -11

141. 8 and 14

142. $\frac{1}{6}$ and $-\frac{2}{5}$

Review

In Exercises 143–146, identify the rule of algebra being demonstrated.

143. $(10x)y = 10(xy)$

144. $-4(x - 3) = -4x + 12$

145. $7x^4 + (-7x^4) = 0$

146. $(x + 4) + x^3 = x + (4 + x^3)$

In Exercises 147–150, find the product.

147. $(x + 3)(x - 6)$

148. $(x - 8)(x - 1)$

149. $(x + 4)(x^2 - x + 2)$

150. $(x + 9)(x^2 - 6x + 4)$

1.5 Complex Numbers

▶ What you should learn

- How to use the imaginary unit i to write complex numbers
- How to add, subtract, and multiply complex numbers
- How to use complex conjugates to write the quotient of two complex numbers in standard form
- How to find complex solutions of quadratic equations

▶ Why you should learn it

You can use complex numbers to model and solve real-life problems in electronics. For instance, in Exercise 84 on page 129, you will learn how to use complex numbers to find the impedance of an electrical circuit.

Chris Hamilton/Corbis

The Imaginary Unit i

In Section 1.4, you learned that some quadratic equations have no real solutions. For instance, the quadratic equation

$$x^2 + 1 = 0 \qquad \text{Equation with no real solution}$$

has no real solution because there is no real number x that can be squared to produce -1. To overcome this deficiency, mathematicians created an expanded system of numbers using the **imaginary unit i,** defined as

$$i = \sqrt{-1} \qquad \text{Imaginary unit}$$

where $i^2 = -1$. By adding real numbers to real multiples of this imaginary unit, the set of **complex numbers** is obtained. Each complex number can be written in the **standard form $a + bi$.** The real number a is called the **real part** of the **complex number $a + bi$,** and the number bi (where b is a real number) is called the **imaginary part** of the complex number.

Definition of a Complex Number

If a and b are real numbers, the number $a + bi$ is a **complex number,** and it is said to be written in **standard form.** If $b = 0$, the number $a + bi = a$ is a real number. If $b \neq 0$, the number $a + bi$ is called an **imaginary number.** A number of the form bi, where $b \neq 0$, is called a **pure imaginary number.**

The set of real numbers is a subset of the set of complex numbers, as shown in Figure 1.23. This is true because every real number a can be written as a complex number using $b = 0$. That is, for every real number a, you can write $a = a + 0i$.

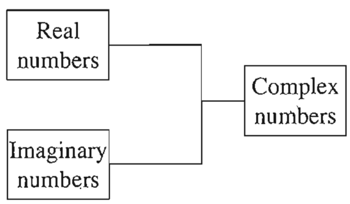


FIGURE 1.23

Equality of Complex Numbers

Two complex numbers $a + bi$ and $c + di$, written in standard form, are equal to each other

$$a + bi = c + di \qquad \text{Equality of two complex numbers}$$

if and only if $a = c$ and $b = d$.

Operations with Complex Numbers

To add (or subtract) two complex numbers, you add (or subtract) the real and imaginary parts of the numbers separately.

Addition and Subtraction of Complex Numbers

If $a + bi$ and $c + di$ are two complex numbers written in standard form, their sum and difference are defined as follows.

Sum: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Difference: $(a + bi) - (c + di) = (a - c) + (b - d)i$

The **additive identity** in the complex number system is zero (the same as in the real number system). Furthermore, the **additive inverse** of the complex number $a + bi$ is

$-(a + bi) = -a - bi.$ Additive inverse

So, you have

$(a + bi) + (-a - bi) = 0 + 0i = 0.$

Example 1 ▶ Adding and Subtracting Complex Numbers

a. $(4 + 7i) + (1 - 6i) = 4 + 7i + 1 - 6i$ Remove parentheses.

$= (4 + 1) + (7i - 6i)$ Group like terms.

$= 5 + i$ Write in standard form.

b. $(1 + 2i) - (4 + 2i) = 1 + 2i - 4 - 2i$ Remove parentheses.

$= (1 - 4) + (2i - 2i)$ Group like terms.

$= -3 + 0$ Simplify.

$= -3$ Write in standard form.

c. $3i - (-2 + 3i) - (2 + 5i) = 3i + 2 - 3i - 2 - 5i$

$= (2 - 2) + (3i - 3i - 5i)$

$= 0 - 5i$

$= -5i$

d. $(3 + 2i) + (4 - i) - (7 + i) = 3 + 2i + 4 - i - 7 - i$

$= (3 + 4 - 7) + (2i - i - i)$

$= 0 + 0i$

$= 0$

Note in Examples 1(b) and 1(d) that the sum of two complex numbers can be a real number.

Exploration

Complete the following.

$i^1 = i$	$i^7 =$
$i^2 = -1$	$i^8 =$
$i^3 = -i$	$i^9 =$
$i^4 = 1$	$i^{10} =$
$i^5 =$	$i^{11} =$
$i^6 =$	$i^{12} =$

What pattern do you see? Write a brief description of how you would find i raised to any positive integer power.

Many of the properties of real numbers are valid for complex numbers as well. Here are some examples.

Associative Properties of Addition and Multiplication

Commutative Properties of Addition and Multiplication

Distributive Property of Multiplication Over Addition

Notice below how these properties are used when two complex numbers are multiplied.

$$(a + bi)(c + di) = a(c + di) + bi(c + di) \quad \text{Distributive Property}$$
$$= ac + (ad)i + (bc)i + (bd)i^2 \quad \text{Distributive Property}$$
$$= ac + (ad)i + (bc)i + (bd)(-1) \quad i^2 = -1$$
$$= ac - bd + (ad)i + (bc)i \quad \text{Commutative Property}$$
$$= (ac - bd) + (ad + bc)i \quad \text{Associative Property}$$

Rather than trying to memorize this multiplication rule, you should simply remember how the Distributive Property is used to multiply two complex numbers.

Example 2 ▶ Multiplying Complex Numbers

a. $4(-2 + 3i) = 4(-2) + 4(3i)$ — Distributive Property

$= -8 + 12i$ — Simplify.

b. $(2 - i)(4 + 3i) = 2(4 + 3i) - i(4 + 3i)$ — Distributive Property

$= 8 + 6i - 4i - 3i^2$ — Distributive Property

$= 8 + 6i - 4i - 3(-1)$ — $i^2 = -1$

$= (8 + 3) + (6i - 4i)$ — Group like terms.

$= 11 + 2i$ — Write in standard form.

c. $(3 + 2i)(3 - 2i) = 3(3 - 2i) + 2i(3 - 2i)$ — Distributive Property

$= 9 - 6i + 6i - 4i^2$ — Distributive Property

$= 9 - 6i + 6i - 4(-1)$ — $i^2 = -1$

$= 9 + 4$ — Simplify.

$= 13$ — Write in standard form.

d. $(3 + 2i)^2 = (3 + 2i)(3 + 2i)$ — Square of a binomial

$= 3(3 + 2i) + 2i(3 + 2i)$ — Distributive Property

$= 9 + 6i + 6i + 4i^2$ — Distributive Property

$= 9 + 6i + 6i + 4(-1)$ — $i^2 = -1$

$= 9 + 12i - 4$ — Simplify.

$= 5 + 12i$ — Write in standard form.

Complex Conjugates

Notice in Example 2(c) that the product of two complex numbers can be a real number. This occurs with pairs of complex numbers of the form $a + bi$ and $a - bi$, called **complex conjugates.**

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - abi + abi - b^2i^2\\ &= a^2 - b^2(-1)\\ &= a^2 + b^2\end{aligned}$$

Example 3 ▶ Multiplying Conjugates

Multiply each complex number by its complex conjugate.

a. $1 + i$ **b.** $4 - 3i$

Solution

a. The complex conjugate of $1 + i$ is $1 - i$.

$$(1 + i)(1 - i) = 1^2 - i^2 = 1 - (-1) = 2$$

b. The complex conjugate of $4 - 3i$ is $4 + 3i$.

$$(4 - 3i)(4 + 3i) = 4^2 - (3i)^2 = 16 - 9i^2 = 16 - 9(-1) = 25$$

To write the quotient of $a + bi$ and $c + di$ in standard form, where c and d are not both zero, multiply the numerator and denominator by the complex conjugate of the *denominator* to obtain

$$\begin{aligned}\frac{a + bi}{c + di} &= \frac{a + bi}{c + di}\left(\frac{c - di}{c - di}\right)\\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}. \qquad \text{Standard form}\end{aligned}$$

Example 4 ▶ Writing a Quotient of Complex Numbers in Standard Form

$$\begin{aligned}\frac{2 + 3i}{4 - 2i} &= \frac{2 + 3i}{4 - 2i}\left(\frac{4 + 2i}{4 + 2i}\right) && \text{Multiply numerator and denominator by complex conjugate of denominator.}\\ &= \frac{8 + 4i + 12i + 6i^2}{16 - 4i^2} && \text{Expand.}\\ &= \frac{8 - 6 + 16i}{16 + 4} && i^2 = -1\\ &= \frac{2 + 16i}{20} && \text{Simplify.}\\ &= \frac{1}{10} + \frac{4}{5}i && \text{Write in standard form.}\end{aligned}$$

Complex Solutions of Quadratic Equations

When using the Quadratic Formula to solve a quadratic equation, you often obtain a result such as $\sqrt{-3}$, which you know is not a real number. By factoring out $i = \sqrt{-1}$, you can write this number in standard form.

$$\sqrt{-3} = \sqrt{3(-1)} = \sqrt{3}\sqrt{-1} = \sqrt{3}i$$

The number $\sqrt{3}i$ is called the *principal square root* of -3.

STUDY TIP

The definition of principal square root uses the rule

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

for $a > 0$ and $b < 0$. This rule is not valid if *both* a and b are negative. For example,

$$\begin{aligned}\sqrt{-5}\sqrt{-5} &= \sqrt{5(-1)}\sqrt{5(-1)}\\ &= \sqrt{5}i\sqrt{5}i\\ &= \sqrt{25}i^2\\ &= 5i^2 = -5\end{aligned}$$

whereas

$$\sqrt{(-5)(-5)} = \sqrt{25} = 5.$$

To avoid problems with square roots of negative numbers, be sure to convert to standard form *before* multiplying.

Principal Square Root of a Negative Number

If a is a positive number, the **principal square root** of the negative number $-a$ is defined as

$$\sqrt{-a} = \sqrt{a}i.$$

Example 5 ▶ Writing Complex Numbers in Standard Form

a. $\sqrt{-3}\sqrt{-12} = \sqrt{3}i\sqrt{12}i = \sqrt{36}i^2 = 6(-1) = -6$

b. $\sqrt{-48} - \sqrt{-27} = \sqrt{48}i - \sqrt{27}i = 4\sqrt{3}i - 3\sqrt{3}i = \sqrt{3}i$

c.
$$\begin{aligned}(-1 + \sqrt{-3})^2 &= (-1 + \sqrt{3}i)^2\\ &= (-1)^2 - 2\sqrt{3}i + (\sqrt{3})^2(i^2)\\ &= 1 - 2\sqrt{3}i + 3(-1)\\ &= -2 - 2\sqrt{3}i\end{aligned}$$

Example 6 ▶ Complex Solutions of a Quadratic Equation

Solve (a) $x^2 + 4 = 0$ and (b) $3x^2 - 2x + 5 = 0$.

Solution

a. $x^2 + 4 = 0$ — Write original equation.

$x^2 = -4$ — Subtract 4 from each side.

$x = \pm 2i$ — Extract square roots.

b. $3x^2 - 2x + 5 = 0$ — Write original equation.

$x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)}$ — Quadratic Formula

$= \dfrac{2 \pm \sqrt{-56}}{6}$ — Simplify.

$= \dfrac{2 \pm 2\sqrt{14}i}{6}$ — Write $\sqrt{-56}$ in standard form.

$= \dfrac{1}{3} \pm \dfrac{\sqrt{14}}{3}i$ — Write in standard form.

1.5 Exercises

In Exercises 1–4, find real numbers a and b such that the equation is true.

1. $a + bi = -10 + 6i$
2. $a + bi = 13 + 4i$
3. $(a - 1) + (b + 3)i = 5 + 8i$
4. $(a + 6) + 2bi = 6 - 5i$

In Exercises 5–16, write the complex number in standard form.

5. $4 + \sqrt{-9}$
6. $3 + \sqrt{-16}$
7. $2 - \sqrt{-27}$
8. $1 + \sqrt{-8}$
9. $\sqrt{-75}$
10. $\sqrt{-4}$
11. 8
12. 45
13. $-6i + i^2$
14. $-4i^2 + 2i$
15. $\sqrt{-0.09}$
16. $\sqrt{-0.0004}$

In Exercises 17–26, perform the addition or subtraction and write the result in standard form.

17. $(5 + i) + (6 - 2i)$
18. $(13 - 2i) + (-5 + 6i)$
19. $(8 - i) - (4 - i)$
20. $(3 + 2i) - (6 + 13i)$
21. $(-2 + \sqrt{-8}) + (5 - \sqrt{-50})$
22. $(8 + \sqrt{-18}) - (4 + 3\sqrt{2}i)$
23. $13i - (14 - 7i)$
24. $22 + (-5 + 8i) + 10i$
25. $-\left(\frac{3}{2} + \frac{5}{2}i\right) + \left(\frac{5}{3} + \frac{11}{3}i\right)$
26. $(1.6 + 3.2i) + (-5.8 + 4.3i)$

In Exercises 27–40, perform the operation and write the result in standard form.

27. $\sqrt{-6} \cdot \sqrt{-2}$
28. $\sqrt{-5} \cdot \sqrt{-10}$
29. $(\sqrt{-10})^2$
30. $(\sqrt{-75})^2$
31. $(1 + i)(3 - 2i)$
32. $(6 - 2i)(2 - 3i)$
33. $6i(5 - 2i)$
34. $-8i(9 + 4i)$
35. $(\sqrt{14} + \sqrt{10}i)(\sqrt{14} - \sqrt{10}i)$
36. $(3 + \sqrt{-5})(7 - \sqrt{-10})$
37. $(4 + 5i)^2$
38. $(2 - 3i)^2$
39. $(2 + 3i)^2 + (2 - 3i)^2$
40. $(1 - 2i)^2 - (1 + 2i)^2$

In Exercises 41–48, write the complex conjugate of the complex number. Then multiply the number by its complex conjugate.

41. $6 + 3i$
42. $7 - 12i$
43. $-1 - \sqrt{5}i$
44. $-3 + \sqrt{2}i$
45. $\sqrt{-20}$
46. $\sqrt{-15}$
47. $\sqrt{8}$
48. $1 + \sqrt{8}$

In Exercises 49–58, write the quotient in standard form.

49. $\dfrac{5}{i}$
50. $-\dfrac{14}{2i}$
51. $\dfrac{2}{4 - 5i}$
52. $\dfrac{5}{1 - i}$
53. $\dfrac{3 + i}{3 - i}$
54. $\dfrac{6 - 7i}{1 - 2i}$
55. $\dfrac{6 - 5i}{i}$
56. $\dfrac{8 + 16i}{2i}$
57. $\dfrac{3i}{(4 - 5i)^2}$
58. $\dfrac{5i}{(2 + 3i)^2}$

In Exercises 59–62, perform the operation and write the result in standard form.

59. $\dfrac{2}{1 + i} - \dfrac{3}{1 - i}$
60. $\dfrac{2i}{2 + i} + \dfrac{5}{2 - i}$
61. $\dfrac{i}{3 - 2i} + \dfrac{2i}{3 + 8i}$
62. $\dfrac{1 + i}{i} - \dfrac{3}{4 - i}$

In Exercises 63–72, use the Quadratic Formula to solve the quadratic equation.

63. $x^2 - 2x + 2 = 0$
64. $x^2 + 6x + 10 = 0$
65. $4x^2 + 16x + 17 = 0$
66. $9x^2 - 6x + 37 = 0$
67. $4x^2 + 16x + 15 = 0$
68. $16t^2 - 4t + 3 = 0$
69. $\frac{3}{2}x^2 - 6x + 9 = 0$
70. $\frac{7}{8}x^2 - \frac{3}{4}x + \frac{5}{16} = 0$
71. $1.4x^2 - 2x - 10 = 0$
72. $4.5x^2 - 3x + 12 = 0$

In Exercises 73–80, simplify the complex number and write it in standard form.

73. $-6i^3 + i^2$

74. $4i^2 - 2i^3$

75. $-5i^5$

76. $(-i)^3$

77. $(\sqrt{-75})^3$

78. $(\sqrt{-2})^6$

79. $\dfrac{1}{i^3}$

80. $\dfrac{1}{(2i)^3}$

81. Cube each complex number.

(a) 2 (b) $-1 + \sqrt{3}i$ (c) $-1 - \sqrt{3}i$

82. Raise each complex number to the fourth power.

(a) 2 (b) -2 (c) $2i$ (d) $-2i$

83. Express each of the powers of i as i, $-i$, 1, or -1.

(a) i^{40} (b) i^{25} (c) i^{50} (d) i^{67}

▶ Model It

84. ***Impedance*** The opposition to current in an electrical circuit is called its impedance. The impedance z in a parallel circuit with two pathways satisfies the equation

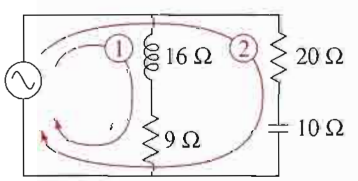
$$\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2}$$

where z_1 is the impedance (in ohms) of pathway 1 and z_2 is the impedance of pathway 2.

(a) The impedance of each pathway in a parallel circuit is found by adding the impedances of all components in the pathway. Use the table to find z_1 and z_2.

(b) Find the impedance z.

	Resistor	Inductor	Capacitor
Symbol	$a\Omega$	$b\Omega$	$c\Omega$
Impedance	a	bi	$-ci$



Synthesis

True or False? **In Exercises 85–87, determine whether the statement is true or false. Justify your answer.**

85. There is no complex number that is equal to its complex conjugate.

86. $-i\sqrt{6}$ is a solution of $x^4 - x^2 + 14 = 56$.

87. $i^{44} + i^{150} - i^{74} - i^{109} + i^{61} = -1$

88. ***Error Analysis*** Describe the error.

~~$\sqrt{-6}\sqrt{-6} = \sqrt{(-6)(-6)} = \sqrt{36} = 6$~~

89. ***Proof*** Prove that the complex conjugate of the product of two complex numbers $a_1 + b_1i$ and $a_2 + b_2i$ is the product of their complex conjugates.

90. ***Proof*** Prove that the complex conjugate of the sum of two complex numbers $a_1 + b_1i$ and $a_2 + b_2i$ is the sum of their complex conjugates.

Review

In Exercises 91–94, perform the operation and write the result in standard form.

91. $(4 + 3x) + (8 - 6x - x^2)$

92. $(x^3 - 3x^2) - (6 - 2x - 4x^2)$

93. $(3x - \frac{1}{2})(x + 4)$

94. $(2x - 5)^2$

In Exercises 95–98, solve the equation and check your solution.

95. $-x - 12 = 19$

96. $8 - 3x = -34$

97. $4(5x - 6) - 3(6x + 1) = 0$

98. $5[x - (3x + 11)] = 20x - 15$

99. ***Volume of an Oblate Spheroid***

Solve for a: $V = \frac{4}{3}\pi a^2 b$

100. ***Newton's Law of Universal Gravitation***

Solve for r: $F = \alpha \dfrac{m_1 m_2}{r^2}$

101. ***Mixture Problem*** A five-liter container contains a mixture with a concentration of 50%. How much of this mixture must be withdrawn and replaced by 100% concentrate to bring the mixture up to 60% concentration?

1.6 Other Types of Equations

▶ What you should learn

- How to solve polynomial equations of degree three or greater
- How to solve equations involving radicals
- How to solve equations involving fractions or absolute values
- How to use polynomial equations and equations involving radicals to model and solve real-life problems

▶ Why you should learn it

Polynomial equations, radical equations, and absolute value equations can be used to model and solve real-life problems. For instance, in Exercise 102 on page 140, a radical equation can be used to model the total cost of a power line project.

Vince Streano/Corbis

Polynomial Equations

In this section you will extend the techniques for solving equations to nonlinear and nonquadratic equations. At this point in the text, you have only four basic methods for solving nonlinear equations—*factoring*, *extracting square roots*, *completing the square*, and the *Quadratic Formula*. So the main goal of this section is to learn to *rewrite* nonlinear equations in a form to which you can apply one of these methods.

Example 1 shows how to use factoring to solve a **polynomial equation,** which is an equation that can be written in the general form

$$a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0 = 0.$$

Example 1 ▶ Solving a Polynomial Equation by Factoring

Solve $3x^4 = 48x^2$.

Solution

First write the polynomial equation in general form with zero on one side, factor the other side, and then set each factor equal to zero and solve.

$3x^4 = 48x^2$		Write original equation.
$3x^4 - 48x^2 = 0$		Write in general form.
$3x^2(x^2 - 16) = 0$		Factor out common factor.
$3x^2(x + 4)(x - 4) = 0$		Write in factored form.
$3x^2 = 0$	⇨ $x = 0$	Set 1st factor equal to 0.
$x + 4 = 0$	⇨ $x = -4$	Set 2nd factor equal to 0.
$x - 4 = 0$	⇨ $x = 4$	Set 3rd factor equal to 0.

You can check these solutions by substituting in the original equation, as follows.

Check

$3(0)^4 = 48(0)^2$	0 checks. ✓
$3(-4)^4 = 48(-4)^2$	−4 checks. ✓
$3(4)^4 = 48(4)^2$	4 checks. ✓

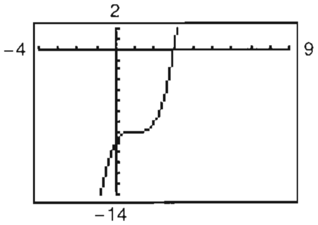
So, you can conclude that the solutions are $x = 0$, $x = -4$, and $x = 4$.

A common mistake that is made in solving an equation such as that in Example 1 is to divide each side of the equation by the variable factor x^2. This loses the solution $x = 0$. When solving an equation, always write the equation in general form, then factor the equation and set each factor equal to zero. Do not divide each side of an equation by a variable factor in an attempt to simplify the equation.

Technology You can use a graphing utility to check graphically the solutions of the equation in Example 2. To do this, graph the equation

$$y = x^3 - 3x^2 + 3x - 9.$$

As shown below, the x-intercept of the graph occurs at the *real* zero of the function, $x = 3$, confirming the result found in Example 2.



Try using a graphing utility to check the solutions found in Example 3.

For a review of factoring special polynomial forms, see Section P.4.

Example 2 ▶ Solving a Polynomial Equation by Factoring

Solve $x^3 - 3x^2 + 3x - 9 = 0$.

Solution

$x^3 - 3x^2 + 3x - 9 = 0$ Write original equation.

$x^2(x - 3) + 3(x - 3) = 0$ Factor by grouping.

$(x - 3)(x^2 + 3) = 0$ Distributive Property

$x - 3 = 0 \Rightarrow x = 3$ Set 1st factor equal to 0.

$x^2 + 3 = 0 \Rightarrow x = \pm\sqrt{3}i$ Set 2nd factor equal to 0.

The solutions are $x = 3$, $x = \sqrt{3}i$, and $x = -\sqrt{3}i$.

Occasionally, mathematical models involve equations that are of quadratic type. In general, an equation is of quadratic type if it can be written in the form

$$au^2 + bu + c = 0$$

where $a \neq 0$ and u is an algebraic expression.

Example 3 ▶ Solving an Equation of Quadratic Type

Solve $x^4 - 3x^2 + 2 = 0$.

Solution

This equation is of quadratic type with $u = x^2$.

$$(x^2)^2 - 3(x^2) + 2 = 0$$

To solve this equation, you can factor the left side of the equation as the product of two second-degree polynomials.

$x^4 - 3x^2 + 2 = 0$ Write original equation.

$\underbrace{(x^2)^2}_{u^2} - \underbrace{3(x^2)}_{3u} + 2 = 0$ Quadratic form

$(x^2 - 1)(x^2 - 2) = 0$ Partially factor.

$(x + 1)(x - 1)(x^2 - 2) = 0$ Factor completely.

$x + 1 = 0 \Rightarrow x = -1$ Set 1st factor equal to 0.

$x - 1 = 0 \Rightarrow x = 1$ Set 2nd factor equal to 0.

$x^2 - 2 = 0 \Rightarrow x = \pm\sqrt{2}$ Set 3rd factor equal to 0.

The solutions are $x = -1$, $x = 1$, $x = \sqrt{2}$, and $x = -\sqrt{2}$. Check these in the original equation.

Equations Involving Radicals

The steps involved in solving the remaining equations in this section will often introduce *extraneous solutions*, as discussed in Section 1.2. Operations such as squaring each side of an equation, raising each side of an equation to a rational power, and multiplying each side of an equation by a variable quantity all can introduce extraneous solutions. So, when you use any of these operations, checking is crucial.

Example 4 ▶ Solving Equations Involving Radicals

a. $\sqrt{2x+7} - x = 2$ — Original equation

$\sqrt{2x+7} = x + 2$ — Isolate radical.

$2x + 7 = x^2 + 4x + 4$ — Square each side.

$0 = x^2 + 2x - 3$ — Write in general form.

$0 = (x+3)(x-1)$ — Factor.

$x + 3 = 0 \implies x = -3$ — Set 1st factor equal to 0.

$x - 1 = 0 \implies x = 1$ — Set 2nd factor equal to 0.

By checking these values, you can determine that the only solution is $x = 1$.

b. $\sqrt{2x-5} - \sqrt{x-3} = 1$ — Original equation

$\sqrt{2x-5} = \sqrt{x-3} + 1$ — Isolate $\sqrt{2x-5}$.

$2x - 5 = x - 3 + 2\sqrt{x-3} + 1$ — Square each side.

$2x - 5 = x - 2 + 2\sqrt{x-3}$ — Combine like terms.

$x - 3 = 2\sqrt{x-3}$ — Isolate $2\sqrt{x-3}$.

$x^2 - 6x + 9 = 4(x-3)$ — Square each side.

$x^2 - 10x + 21 = 0$ — Write in general form.

$(x-3)(x-7) = 0$ — Factor.

$x - 3 = 0 \implies x = 3$ — Set 1st factor equal to 0.

$x - 7 = 0 \implies x = 7$ — Set 2nd factor equal to 0.

The solutions are $x = 3$ and $x = 7$. Check these in the original equation.

STUDY TIP

The essential operations in Example 4 are isolating the square root and squaring each side. In Example 5, this is equivalent to isolating the factor with the rational exponent and raising each side to the *reciprocal power*.

Example 5 ▶ Solving an Equation Involving a Rational Exponent

$(x-4)^{2/3} = 25$ — Original equation

$x - 4 = 25^{3/2}$ — Raise each side to the $\frac{3}{2}$ power.

$x - 4 = 125$ — Simplify.

$x = 129$ — Add 4 to each side.

The solution is $x = 129$. Check this in the original equation.

Equations with Fractions or Absolute Values

To solve an equation involving fractions, multiply each side of the equation by the least common denominator (LCD) of all terms in the equation. This procedure will "clear the equation of fractions." For instance, in the equation

$$\frac{2}{x^2 + 1} + \frac{1}{x} = \frac{2}{x}$$

you can multiply each side of the equation by $x(x^2 + 1)$. Try doing this and solve the resulting equation. You should obtain one solution: $x = 1$.

Example 6 ▶ Solving an Equation Involving Fractions

Solve $\frac{2}{x} = \frac{3}{x - 2} - 1$.

Solution

For this equation, the least common denominator of the three terms is $x(x - 2)$, so you begin by multiplying each term of the equation by this expression.

$$\frac{2}{x} = \frac{3}{x - 2} - 1$$ Write original equation.

$$x(x - 2)\frac{2}{x} = x(x - 2)\frac{3}{x - 2} - x(x - 2)(1)$$ Multiply each term by the LCD.

$$2(x - 2) = 3x - x(x - 2)$$ Simplify.

$$2x - 4 = -x^2 + 5x$$ Simplify.

$$x^2 - 3x - 4 = 0$$ Write in general form.

$$(x - 4)(x + 1) = 0$$ Factor.

$$x - 4 = 0 \quad \Rightarrow \quad x = 4$$ Set 1st factor equal to 0.

$$x + 1 = 0 \quad \Rightarrow \quad x = -1$$ Set 2nd factor equal to 0.

Check $x = 4$

$$\frac{2}{x} = \frac{3}{x - 2} - 1$$

$$\frac{2}{4} \stackrel{?}{=} \frac{3}{4 - 2} - 1$$

$$\frac{1}{2} \stackrel{?}{=} \frac{3}{2} - 1$$

$$\frac{1}{2} = \frac{1}{2} \checkmark$$

Check $x = -1$

$$\frac{2}{x} = \frac{3}{x - 2} - 1$$

$$\frac{2}{-1} \stackrel{?}{=} \frac{3}{-1 - 2} - 1$$

$$-2 \stackrel{?}{=} -1 - 1$$

$$-2 = -2 \checkmark$$

So, the solutions are $x = 4$ and $x = -1$.

To solve an equation involving an absolute value, remember that the expression inside the absolute value signs can be positive or negative. This results in *two* separate equations, each of which must be solved. For instance, the equation

$$|x - 2| = 3$$

results in the two equations $x - 2 = 3$ and $-(x - 2) = 3$, which implies that the equation has two solutions: $x = 5$ and $x = -1$.

Example 7 ▶ Solving an Equation Involving Absolute Value

Solve $|x^2 - 3x| = -4x + 6$.

Solution

Because the variable expression inside the absolute value signs can be positive or negative, you must solve the following two equations.

First Equation

$x^2 - 3x = -4x + 6$ Use positive expression.

$x^2 + x - 6 = 0$ Write in general form.

$(x + 3)(x - 2) = 0$ Factor.

$x + 3 = 0 \Rightarrow x = -3$ Set 1st factor equal to 0.

$x - 2 = 0 \Rightarrow x = 2$ Set 2nd factor equal to 0.

Second Equation

$-(x^2 - 3x) = -4x + 6$ Use negative expression.

$x^2 - 7x + 6 = 0$ Write in general form.

$(x - 1)(x - 6) = 0$ Factor.

$x - 1 = 0 \Rightarrow x = 1$ Set 1st factor equal to 0.

$x - 6 = 0 \Rightarrow x = 6$ Set 2nd factor equal to 0.

Check

$|(-3)^2 - 3(-3)| \stackrel{?}{=} -4(-3) + 6$ Substitute -3 for x.

$18 = 18$ -3 checks. ✓

$|(2)^2 - 3(2)| \stackrel{?}{=} -4(2) + 6$ Substitute 2 for x.

$2 \neq -2$ 2 does not check.

$|(1)^2 - 3(1)| \stackrel{?}{=} -4(1) + 6$ Substitute 1 for x.

$2 = 2$ 1 checks. ✓

$|(6)^2 - 3(6)| \stackrel{?}{=} -4(6) + 6$ Substitute 6 for x.

$18 \neq -18$ 6 does not check.

The solutions are $x = -3$ and $x = 1$.

Applications

It would be impossible to categorize the many different types of applications that involve nonlinear and nonquadratic models. However, from the few examples and exercises that are given, you will gain some appreciation for the variety of applications that can occur.

Example 8 ▶ Reduced Rates

A ski club chartered a bus for a ski trip at a cost of \$480. In an attempt to lower the bus fare per skier, the club invited nonmembers to go along. After five nonmembers joined the trip, the fare per skier decreased by \$4.80. How many club members are going on the trip?

Solution

Begin the solution by creating a verbal model and assigning labels.

Verbal Model: Cost per skier · Number of skiers = Cost of trip

Labels:
Cost of trip $= 480$ (dollars)
Number of ski club members $= x$ (people)
Number of skiers $= x + 5$ (people)
Original cost per member $= \dfrac{480}{x}$ (dollars per person)
Cost per skier $= \dfrac{480}{x} - 4.80$ (dollars per person)

Equation:
$$\left(\frac{480}{x} - 4.80\right)(x + 5) = 480$$

$$\left(\frac{480 - 4.8x}{x}\right)(x + 5) = 480 \qquad \text{Write } \left(\frac{480}{x} - 4.80\right) \text{ as a fraction.}$$

$$(480 - 4.8x)(x + 5) = 480x \qquad \text{Multiply each side by } x.$$

$$480x + 2400 - 4.8x^2 - 24x = 480x \qquad \text{Multiply.}$$

$$-4.8x^2 - 24x + 2400 = 0 \qquad \text{Subtract } 480x \text{ from each side.}$$

$$x^2 + 5x - 500 = 0 \qquad \text{Divide each side by } -4.8.$$

$$(x + 25)(x - 20) = 0 \qquad \text{Factor.}$$

$$x + 25 = 0 \implies x = -25$$

$$x - 20 = 0 \implies x = 20$$

Choosing the positive value of x, you can conclude that 20 ski club members are going on the trip. Check this in the original statement of the problem, as follows.

$$\left(\frac{480}{20} - 4.80\right)(20 + 5) \stackrel{?}{=} 480 \qquad \text{Substitute 20 for } x.$$

$$(24 - 4.80)25 \stackrel{?}{=} 480 \qquad \text{Simplify.}$$

$$480 = 480 \qquad \text{20 checks. ✓}$$

Interest in a savings account is calculated by one of three basic methods: simple interest, interest compounded n times per year, and interest compounded continuously. The next example uses the formula for interest that is compounded n times per year.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

In this formula, A is the balance in the account, P is the principal (or original deposit), r is the annual interest rate (in decimal form), n is the number of compoundings per year, and t is the time in years. In Chapter 5, you will study a derivation of the formula above for interest compounded continuously.

Example 9 ▶ Compound Interest

When your cousin was born, your grandparents deposited $5000 in a long-term investment in which the interest was compounded quarterly. Today, on your cousin's 25th birthday, the value of the investment is $25,062.59. What is the annual interest rate for this investment?

Solution

Formula: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

Labels:

Balance $= A = 25{,}062.59$	(dollars)
Principal $= P = 5000$	(dollars)
Time $= t = 25$	(years)
Compoundings per year $= n = 4$	(compoundings per year)
Annual interest rate $= r$	(percent in decimal form)

Equation:

$$25{,}062.59 = 5000\left(1 + \frac{r}{4}\right)^{4(25)}$$

$$\frac{25{,}062.59}{5000} = \left(1 + \frac{r}{4}\right)^{100}$$ Divide each side by 5000.

$$5.0125 \approx \left(1 + \frac{r}{4}\right)^{100}$$ Use a calculator.

$$(5.0125)^{1/100} = 1 + \frac{r}{4}$$ Raise each side to reciprocal power.

$$1.01625 \approx 1 + \frac{r}{4}$$ Use a calculator.

$$0.01625 = \frac{r}{4}$$ Subtract 1 from each side.

$$0.065 = r$$ Multiply each side by 4.

The annual interest rate is about 0.065, or 6.5%. Check this in the original statement of the problem.

1.6 Exercises

In Exercises 1–24, find all solutions of the equation. Check your solutions in the original equation.

1. $4x^4 - 18x^2 = 0$
2. $20x^3 - 125x = 0$
3. $x^4 - 81 = 0$
4. $x^6 - 64 = 0$
5. $x^3 + 216 = 0$
6. $27x^3 - 512 = 0$
7. $5x^3 + 30x^2 + 45x = 0$
8. $9x^4 - 24x^3 + 16x^2 = 0$
9. $x^3 - 3x^2 - x + 3 = 0$
10. $x^3 + 2x^2 + 3x + 6 = 0$
11. $x^4 - x^3 + x - 1 = 0$
12. $x^4 + 2x^3 - 8x - 16 = 0$
13. $x^4 - 4x^2 + 3 = 0$
14. $x^4 + 5x^2 - 36 = 0$
15. $4x^4 - 65x^2 + 16 = 0$
16. $36t^4 + 29t^2 - 7 = 0$
17. $x^6 + 7x^3 - 8 = 0$
18. $x^6 + 3x^3 + 2 = 0$
19. $\frac{1}{x^2} + \frac{8}{x} + 15 = 0$
20. $6\left(\frac{x}{x+1}\right)^2 + 5\left(\frac{x}{x+1}\right) - 6 = 0$
21. $2x + 9\sqrt{x} = 5$
22. $6x - 7\sqrt{x} - 3 = 0$
23. $3x^{1/3} + 2x^{2/3} = 5$
24. $9t^{2/3} + 24t^{1/3} + 16 = 0$

Graphical Analysis In Exercises 25–28, (a) use a graphing utility to graph the equation; (b) use the graph to approximate any x-intercepts of the graph; (c) set $y = 0$ and solve the resulting equation; and (d) compare the result of part (c) with the x-intercepts of the graph.

25. $y = x^3 - 2x^2 - 3x$
26. $y = 2x^4 - 15x^3 + 18x^2$
27. $y = x^4 - 10x^2 + 9$
28. $y = x^4 - 29x^2 + 100$

In Exercises 29–52, find all solutions of the equation. Check your solutions in the original equation.

29. $\sqrt{2x} - 10 = 0$
30. $4\sqrt{x} - 3 = 0$
31. $\sqrt{x - 10} - 4 = 0$
32. $\sqrt{5 - x} - 3 = 0$
33. $\sqrt[3]{2x + 5} + 3 = 0$
34. $\sqrt[3]{3x + 1} - 5 = 0$
35. $-\sqrt{26 - 11x} + 4 = x$
36. $x + \sqrt{31 - 9x} = 5$
37. $\sqrt{x + 1} = \sqrt{3x + 1}$
38. $\sqrt{x + 5} = \sqrt{x - 5}$
39. $\sqrt{x} - \sqrt{x - 5} = 1$
40. $\sqrt{x} + \sqrt{x - 20} = 10$
41. $\sqrt{x + 5} + \sqrt{x - 5} = 10$
42. $2\sqrt{x + 1} - \sqrt{2x + 3} = 1$
43. $\sqrt{x + 2} - \sqrt{2x - 3} = -1$
44. $4\sqrt{x - 3} - \sqrt{6x - 17} = 3$
45. $(x - 5)^{3/2} = 8$
46. $(x + 3)^{3/2} = 8$
47. $(x + 3)^{2/3} = 8$
48. $(x + 2)^{2/3} = 9$
49. $(x^2 - 5)^{3/2} = 27$
50. $(x^2 - x - 22)^{3/2} = 27$
51. $3x(x - 1)^{1/2} + 2(x - 1)^{3/2} = 0$
52. $4x^2(x - 1)^{1/3} + 6x(x - 1)^{4/3} = 0$

Graphical Analysis In Exercises 53–56, (a) use a graphing utility to graph the equation; (b) use the graph to approximate any x-intercepts of the graph; (c) set $y = 0$ and solve the resulting equation; and (d) compare the result of part (c) with the x-intercepts of the graph.

53. $y = \sqrt{11x - 30} - x$
54. $y = 2x - \sqrt{15 - 4x}$
55. $y = \sqrt{7x + 36} - \sqrt{5x + 16} - 2$
56. $y = 3\sqrt{x} - \frac{4}{\sqrt{x}} - 4$

In Exercises 57–70, find all solutions of the equation. Check your solutions in the original equation.

57. $x = \frac{3}{x} + \frac{1}{2}$

58. $\frac{4}{x} - \frac{5}{3} = \frac{x}{6}$

59. $\frac{1}{x} - \frac{1}{x+1} = 3$

60. $\frac{4}{x+1} - \frac{3}{x+2} = 1$

61. $\frac{20-x}{x} = x$

62. $4x + 1 = \frac{3}{x}$

63. $\frac{x}{x^2-4} + \frac{1}{x+2} = 3$

64. $\frac{x+1}{3} - \frac{x+1}{x+2} = 0$

65. $|2x - 1| = 5$

66. $|3x + 2| = 7$

67. $|x| = x^2 + x - 3$

68. $|x^2 + 6x| = 3x + 18$

69. $|x + 1| = x^2 - 5$

70. $|x - 10| = x^2 - 10x$

Graphical Analysis **In Exercises 71–74, (a) use a graphing utility to graph the equation; (b) use the graph to approximate any x-intercepts of the graph; (c) set $y = 0$ and solve the resulting equation; and (d) compare the result of part (c) with the x-intercepts of the graph.**

71. $y = \frac{1}{x} - \frac{4}{x-1} - 1$

72. $y = x + \frac{9}{x+1} - 5$

73. $y = |x + 1| - 2$

74. $y = |x - 2| - 3$

In Exercises 75–78, find the real solutions of the equation analytically. (Round your answers to three decimal places.)

75. $3.2x^4 - 1.5x^2 - 2.1 = 0$

76. $7.08x^6 + 4.15x^3 - 9.6 = 0$

77. $1.8x - 6\sqrt{x} - 5.6 = 0$

78. $4x^{2/3} + 8x^{1/3} + 3.6 = 0$

Think About It **In Exercises 79–86, find an equation that has the given solutions. (There are many correct answers.)**

79. $-2, 5$

80. $0, 3, 5$

81. $-\frac{7}{3}, \frac{6}{7}$

82. $-\frac{1}{8}, -\frac{4}{5}$

83. $\sqrt{3}, -\sqrt{3}, 4$

84. $2\sqrt{7}, -\sqrt{7}$

85. $-1, 1, i, -i$

86. $4i, -4i, 6, -6$

87. ***Chartering a Bus*** A college charters a bus for \$1700 to take a group to a museum. When six more students join the trip, the cost per student drops by \$7.50. How many students were in the original group?

88. ***Renting an Apartment*** Three students are planning to rent an apartment for a year and share equally in the cost. By adding a fourth person, each person could save \$75 a month. How much is the monthly rent?

89. ***Airspeed*** An airline runs a commuter flight between Portland, Oregon and Seattle, Washington, which are 145 miles apart. If the average speed of the plane could be increased by 40 miles per hour, the travel time would be decreased by 12 minutes. What airspeed is required to obtain this decrease in travel time?

90. ***Average Speed*** A family drove 1080 miles to their vacation lodge. Because of increased traffic density, their average speed on the return trip was decreased by 6 miles per hour and the trip took $2\frac{1}{2}$ hours longer. Determine their average speed on the way to the lodge.

91. ***Mutual Funds*** A deposit of \$2500 in a mutual fund reaches a balance of \$3052.49 after 5 years. What annual interest rate on a certificate of deposit compounded monthly would yield an equivalent return?

92. ***Mutual Funds*** A sales representative for a mutual funds company describes a "guaranteed investment fund" that the company is offering to new investors. You are told that if you deposit \$10,000 in the fund you will be guaranteed a return of at least \$25,000 after 20 years. (Assume the interest is compounded quarterly.)

(a) What is the annual interest rate if the investment only meets the minimum guaranteed amount?

(b) After 20 years, you receive \$32,000. What is the annual interest rate?

93. ***Saturated Steam*** The temperature T (in degrees Fahrenheit) of saturated steam increases as pressure increases. This relationship is approximated by the model

$$T = 75.82 - 2.11x + 43.51\sqrt{x}, \quad 5 \le x \le 40$$

where x is the absolute pressure (in pounds per square inch).

(a) Use the model to complete the table.

Absolute pressure, x	Temperature, T
5	
10	
15	
20	
25	
30	
35	
40	

(b) The temperature of steam at sea level is 212°F. Use the table in part (a) to approximate the absolute pressure at this temperature.

(c) Solve part (b) algebraically.

(d) Use a graphing utility to verify your solutions for part (b) and part (c).

94. ***Airline Passengers*** An airline offers daily flights between Chicago and Denver. The total monthly cost C (in millions of dollars) of these flights is $C = \sqrt{0.2x + 1}$, where x is the number of passengers (in thousands). The total cost of the flights for June is 2.5 million dollars. How many passengers flew in June?

95. ***Demand*** The demand equation for a video game is modeled by $p = 40 - \sqrt{0.01x + 1}$, where x is the number of units demanded per day and p is the price per unit. Approximate the demand when the price is \$37.55.

96. ***Demand*** The demand equation for a coffee maker is modeled by $p = 40 - \sqrt{0.0001x + 1}$, where x is the number of units demanded per day and p is the price per unit. Approximate the demand when the price is \$34.70.

97. ***Baseball*** A baseball diamond has the shape of a square in which the distance from home plate to second base is approximately $127\frac{1}{2}$ feet. Approximate the distance between the bases.

98. ***Meteorology*** A meteorologist is positioned 100 feet from the point where a weather balloon is launched. When the balloon is at height h, the distance d (in feet) between the meteorologist and the balloon is $d = \sqrt{100^2 + h^2}$.

(a) Use a graphing utility to graph the equation. Use the *trace* feature to approximate the value of h when $d = 200$.

(b) Complete the table. Use the table to approximate the value of h when $d = 200$.

h	160	165	170	175	180	185
d						

(c) Find h algebraically when $d = 200$.

(d) Compare the results of each method. In each case, what information did you gain that wasn't apparent in another solution method?

99. ***Geometry*** You construct a cone with a base radius of 8 inches. The surface area S of the cone can be represented by the equation

$$S = 8\pi\sqrt{64 + h^2}$$

where h is the height of the cone.

(a) Use a graphing utility to graph the equation. Use the *trace* feature to approximate the value of h when $S = 350$ square inches.

(b) Complete the table. Use the table to approximate the value of h when $S = 350$.

h	8	9	10	11	12	13
S						

(c) Find h algebraically when $S = 350$.

(d) Compare the results of each method. In each case, what information did you gain that wasn't apparent in another solution method?

100. ***Labor*** Working together, two people can complete a task in 8 hours. Working alone, one person takes 2 hours longer than the other to complete the task. How long would it take for each person to complete the task?

101. ***Labor*** Working together, two people can complete a task in 12 hours. Working alone, one person takes 3 hours longer than the other to complete the task. How long would it take for each person to complete the task?

▶ Model It

102. ***Power Line*** A power station is on one side of a river that is $\frac{3}{4}$ mile wide, and a factory is 8 miles downstream on the other side of the river, as shown in the figure. It costs \$24 per foot to run power lines overland and \$30 per foot to run them underwater.

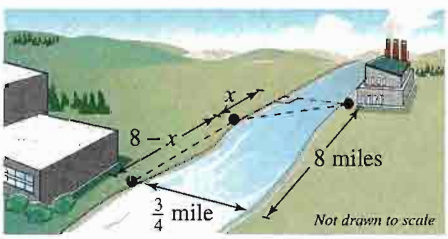
(a) Write the total cost C of the project as a function of x (see figure).

(b) Find the total cost when $x = 3$.

(c) Find the length x when $C = \$1{,}098{,}662.40$.

(d) Use a graphing utility to graph the function from part (a).

(e) Use your graph from part (d) to find the value of x that minimizes the cost.



In Exercises 103 and 104, solve for the indicated variable.

103. ***A Person's Tangential Speed in a Rotor***

Solve for g: $v = \sqrt{\dfrac{gR}{\mu s}}$

104. ***Inductance***

Solve for Q: $i = \pm\sqrt{\dfrac{1}{LC}}\sqrt{Q^2 - q}$

Synthesis

True or False? **In Exercises 105 and 106, determine whether the statement is true or false. Justify your answer.**

105. An equation can never have more than one extraneous solution.

106. When solving an absolute value equation, you will always have to check more than one solution.

In Exercises 107 and 108, find x such that the distance between the given points is 13. Explain your results.

107. $(1, 2), (x, -10)$

108. $(-8, 0), (x, 5)$

In Exercises 109 and 110, find y such that the distance between the given points is 17. Explain your results.

109. $(0, 0), (8, y)$

110. $(-8, 4), (7, y)$

In Exercises 111 and 112, consider an equation of the form $x + |x - a| = b$, where a and b are constants.

111. Find a and b when the solution to the equation is $x = 9$. (There are many correct answers.)

112. ***Writing*** Write a short paragraph listing the steps required to solve this equation involving absolute values.

In Exercises 113 and 114, consider an equation of the form $x + \sqrt{x - a} = b$, where a and b are constants.

113. Find a and b when the solution to the equation is $x = 20$. (There are many correct answers.)

114. ***Writing*** Write a short paragraph listing the steps required to solve this equation involving radicals.

Review

In Exercises 115–118, perform the operation and simplify.

115. $\dfrac{8}{3x} + \dfrac{3}{2x}$

116. $\dfrac{2}{x^2 - 4} - \dfrac{1}{x^2 - 3x + 2}$

117. $\dfrac{2}{z + 2} - \left(3 - \dfrac{2}{z}\right)$

118. $25y^2 \div \dfrac{xy}{5}$

In Exercises 119 and 120, find all real solutions of the equation.

119. $x^2 - 22x + 121 = 0$

120. $x(x - 20) + 3(x - 20) = 0$

1.7 Linear Inequalities in One Variable

▶ **What you should learn**

- How to represent solutions of linear inequalities in one variable.
- How to solve linear inequalities in one variable.
- How to solve inequalities involving absolute values
- How to use inequalities to model and solve real-life problems

▶ **Why you should learn it**

Inequalities can be used to model and solve real-life problems. For instance, in Exercise 98 on page 149, you will use a linear inequality to analyze data about the maximum weight a weightlifter can bench press.

Brian Smith/Stock Boston

Introduction

Simple inequalities were reviewed in Section P.1. There, you used the inequality symbols $<$, $\leq$, $>$, and $\geq$ to compare two numbers and to denote subsets of real numbers. For instance, the simple inequality

$$x \geq 3$$

denotes all real numbers x that are greater than or equal to 3.

In this section you will expand your work with inequalities to include more involved statements such as

$$5x - 7 < 3x + 9$$

and

$$-3 \leq 6x - 1 < 3.$$

As with an equation, you **solve an inequality** in the variable x by finding all values of x for which the inequality is true. Such values are **solutions** and are said to **satisfy** the inequality. The set of all real numbers that are solutions of an inequality is the **solution set** of the inequality. For instance, the solution set of

$$x + 1 < 4$$

is all real numbers that are less than 3.

The set of all points on the real number line that represent the solution set is the **graph of the inequality.** Graphs of many types of inequalities consist of intervals on the real number line. See Section P.1 to review the nine basic types of intervals on the real number line. Note that each type of interval can be classified as *bounded or unbounded.*

Example 1 ▶ Intervals and Inequalities

Write an inequality to represent each interval, and state whether the interval is bounded or unbounded.

a. $(-3, 5]$

b. $(-3, \infty)$

c. $[0, 2]$

d. $(-\infty, \infty)$

Solution

a. $(-3, 5]$ corresponds to $-3 < x \leq 5$. Bounded

b. $(-3, \infty)$ corresponds to $-3 < x$. Unbounded

c. $[0, 2]$ corresponds to $0 \leq x \leq 2$. Bounded

d. $(-\infty, \infty)$ corresponds to $-\infty < x < \infty$. Unbounded

Properties of Inequalities

The procedures for solving linear inequalities in one variable are much like those for solving linear equations. To isolate the variable, you can make use of the **Properties of Inequalities.** These properties are similar to the properties of equality, but there are two important exceptions. When each side of an inequality is multiplied or divided by a negative number, the direction of the inequality symbol must be reversed. Here is an example.

$-2 < 5$	Original inequality
$(-3)(-2) > (-3)(5)$	Multiply each side by -3 and reverse inequality.
$6 > -15$	Simplify.

Two inequalities that have the same solution set are **equivalent.** For instance, the inequalities

$$x + 2 < 5$$

and

$$x < 3$$

are equivalent. To obtain the second inequality from the first, you can subtract 2 from each side of the inequality. The following list describes the operations that can be used to create equivalent inequalities.

Properties of Inequalities

Let a, b, c, and d be real numbers.

1. Transitive Property

$a < b$ and $b < c$ ⇒ $a < c$

2. Addition of Inequalities

$a < b$ and $c < d$ ⇒ $a + c < b + d$

3. Addition of a Constant

$a < b$ ⇒ $a + c < b + c$

4. Multiplication by a Constant

For $c > 0$, $a < b$ ⇒ $ac < bc$

For $c < 0$, $a < b$ ⇒ $ac > bc$

Each of the properties above is true if the symbol $<$ is replaced by $\leq$ and the symbol $>$ is replaced by $\geq$. For instance, another form of the multiplication property would be as follows.

For $c > 0$, $a \leq b$ ⇒ $ac \leq bc$

For $c < 0$, $a \leq b$ ⇒ $ac \geq bc$

Solving a Linear Inequality in One Variable

The simplest type of inequality is a **linear inequality** in one variable. For instance, $2x + 3 > 4$ is a linear inequality in x.

In the following examples, pay special attention to the steps in which the inequality symbol is reversed. Remember that when you multiply or divide by a negative number, you must reverse the inequality symbol.

Example 2 ▶ Solving Linear Inequalities

Solve each inequality.

a. $5x - 7 > 3x + 9$

b. $1 - \frac{3x}{2} \geq x - 4$

Solution

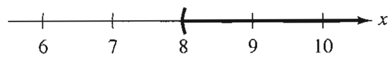
a. $5x - 7 > 3x + 9$ Write original inequality.

$2x - 7 > 9$ Subtract $3x$ from each side.

$2x > 16$ Add 7 to each side.

$x > 8$ Divide each side by 2.

The solution set is all real numbers that are greater than 8, which is denoted by $(8, \infty)$. The graph of this solution set is shown in Figure 1.24.



Solution interval: $(8, \infty)$
FIGURE 1.24

b. $1 - \frac{3x}{2} \geq x - 4$ Write original inequality.

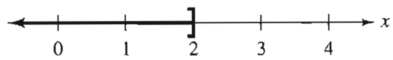
$2 - 3x \geq 2x - 8$ Multiply each side by 2.

$2 - 5x \geq -8$ Subtract $2x$ from each side.

$-5x \geq -10$ Subtract 2 from each side.

$x \leq 2$ Divide each side by -5 and reverse the inequality.

The solution set is all real numbers that are less than or equal to 2, which is denoted by $(-\infty, 2]$. The graph of this solution set is shown in Figure 1.25.



Solution interval: $(-\infty, 2]$
FIGURE 1.25

STUDY TIP

Checking the solution set of an inequality is not as simple as checking the solutions of an equation. You can, however, get an indication of the validity of a solution set by substituting a few convenient values of x.

Sometimes it is possible to write two inequalities as a **double inequality.** For instance, you can write the two inequalities $-4 \le 5x - 2$ and $5x - 2 < 7$ more simply as

$$-4 \le 5x - 2 < 7.$$

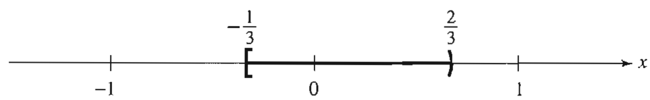
This form allows you to solve the two inequalities together, as demonstrated in Example 3.

Example 3 ▶ Solving a Double Inequality

To solve a double inequality, you can isolate x as the middle term.

$-3 \le 6x - 1 < 3$	Original inequality
$-3 + 1 \le 6x - 1 + 1 < 3 + 1$	Add 1 to each part.
$-2 \le 6x < 4$	Simplify.
$\frac{-2}{6} \le \frac{6x}{6} < \frac{4}{6}$	Divide each part by 6.
$-\frac{1}{3} \le x < \frac{2}{3}$	Simplify.

The solution set is all real numbers that are greater than or equal to $-\frac{1}{3}$ and less than $\frac{2}{3}$, which is denoted by $\left[-\frac{1}{3}, \frac{2}{3}\right)$. The graph of this solution set is shown in Figure 1.26.



Solution interval: $\left[-\frac{1}{3}, \frac{2}{3}\right)$

FIGURE 1.26

The double inequality in Example 3 could have been solved in two parts as follows.

$-3 \le 6x - 1$	and	$6x - 1 < 3$
$-2 \le 6x$		$6x < 4$
$-\frac{1}{3} \le x$		$x < \frac{2}{3}$

The solution set consists of all real numbers that satisfy *both* inequalities. In other words, the solution set is the set of all values of x for which

$$-\frac{1}{3} \le x < \frac{2}{3}.$$

When combining two inequalities to form a double inequality, be sure that the inequalities satisfy the Transitive Property. For instance, it is *incorrect* to combine the inequalities $3 < x$ and $x \le -1$ as $3 < x \le -1$. This "inequality" is wrong because 3 is not less than -1.

Inequalities Involving Absolute Values

Solving an Absolute Value Inequality

Let x be a variable or an algebraic expression and let a be a real number such that $a \geq 0$.

1. The solutions of $|x| < a$ are all values of x that lie between $-a$ and a.

$$|x| < a \quad \text{if and only if} \quad -a < x < a.$$

2. The solutions of $|x| > a$ are all values of x that are less than $-a$ or greater than a.

$$|x| > a \quad \text{if and only if} \quad x < -a \quad \text{or} \quad x > a.$$

These rules are also valid if $<$ is replaced by $\leq$ and $>$ is replaced by $\geq$.

Example 4 ▶ Solving an Absolute Value Inequality

Solve each inequality.

a. $|x - 5| < 2$ **b.** $|x + 3| \geq 7$

Solution

a.

$|x - 5| < 2$ Write original inequality.

$-2 < x - 5 < 2$ Write equivalent inequalities.

$-2 + 5 < x - 5 + 5 < 2 + 5$ Add 5 to each part.

$3 < x < 7$ Simplify.

The solution set is all real numbers that are greater than 3 and less than 7, which is denoted by $(3, 7)$. The graph of this solution set is shown in Figure 1.27.

b.

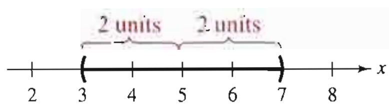
$|x + 3| \geq 7$ Write original inequality.

$x + 3 \leq -7$ or $x + 3 \geq 7$ Write equivalent inequalities.

$x + 3 - 3 \leq -7 - 3$ $x + 3 - 3 \geq 7 - 3$ Subtract 3 from each side.

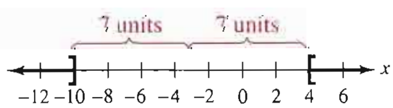
$x \leq -10$ $x \geq 4$ Simplify.

The solution set is all real numbers that are less than or equal to -10 *or* greater than or equal to 4. The interval notation for this solution set is $(-\infty, -10] \cup [4, \infty)$. The symbol $\cup$ is called a *union* symbol and is used to denote the combining of two sets. The graph of this solution set is shown in Figure 1.28.



$|x - 5| < 2$: *Solutions lie inside* $(3, 7)$

FIGURE 1.27



$|x + 3| \geq 7$: *Solutions lie outside* $(-10, 4)$

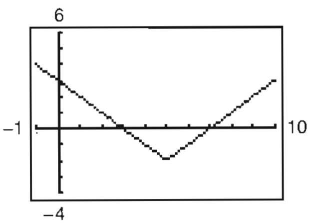
FIGURE 1.28

Technology

A graphing utility can be used to identify the solution set of the graph of an inequality. For instance, to find the solution set of $|x - 5| < 2$ (see Example 4), rewrite the inequality as $|x - 5| - 2 < 0$ and enter

Y1 = abs (X − 5) − 2

and press the graph key. The graph should look like the one shown below.



Notice that the graph is below the x-axis on the interval $(3, 7)$.

STUDY TIP

Note that the graph of the inequality $|x - 5| < 2$ can be described as all real numbers *within* two units of 5, as shown in Figure 1.27.

Applications

The problem-solving plan described in Section 1.3 can be used to model and solve real-life problems that involve inequalities, as illustrated in Example 5.

Example 5 ▶ Comparative Shopping

A subcompact car can be rented from Company A for \$180 per week with no extra charge for mileage. A similar car can be rented from Company B for \$100 per week plus 20 cents for each mile driven. How many miles must you drive in a week in order for the rental fee for Company B to be more than that for Company A?

Solution

Verbal Model: Weekly cost for Company B $>$ Weekly cost for Company A

Labels: Miles driven in one week $= m$ (miles)
Weekly cost for Company A $= 180$ (dollars)
Weekly cost for Company B $= 100 + 0.20m$ (dollars)

Inequality: $100 + 0.2m > 180$

$$0.2m > 80$$

$$m > 400 \text{ miles}$$

If you drive more than 400 miles in a week, Company B costs more.

Example 6 ▶ Accuracy of a Measurement

You go to a candy store to buy chocolates that cost \$9.89 per pound. The scale that is used in the store has a state seal of approval that indicates the scale is accurate to within half an ounce. According to the scale, your purchase weighs one-half pound and costs \$4.95. How much might you have been undercharged or overcharged as a result of inaccuracy in the scale?

Solution

Let x represent the *true* weight of the candy. Because the scale is accurate to within half an ounce (or $\frac{1}{32}$ of a pound), the difference between the exact weight (x) and the scale weight $\left(\frac{1}{2}\right)$ is less than or equal to $\frac{1}{32}$ of a pound. That is, $\left|x - \frac{1}{2}\right| \le \frac{1}{32}$. You can solve this inequality as follows.

$$-\frac{1}{32} \le x - \frac{1}{2} \le \frac{1}{32}$$

$$\frac{15}{32} \le x \le \frac{17}{32}$$

$$0.46875 \le x \le 0.53125$$

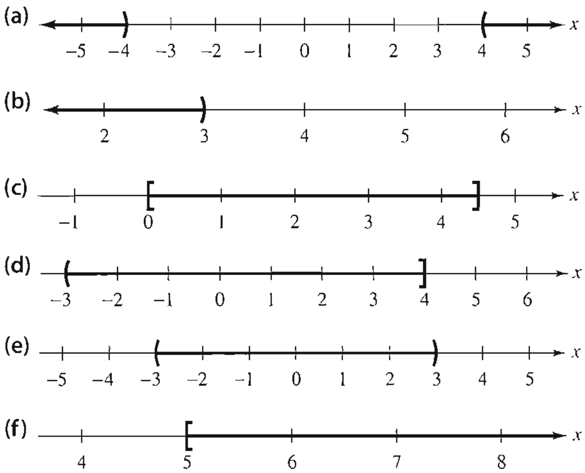
In other words, your "one-half pound" of candy could have weighed as little as 0.46875 pound (which would have cost \$4.64) or as much as 0.53125 pound (which would have cost \$5.25). So, you could have been overcharged by as much as \$0.31 or undercharged by as much as \$0.30.

1.7 Exercises

In Exercises 1–6, write an inequality that represents the interval, and state whether the interval is bounded or unbounded.

1. $[-1, 5]$
2. $(2, 10]$
3. $(11, \infty)$
4. $[-5, \infty)$
5. $(-\infty, -2)$
6. $(-\infty, 7]$

In Exercises 7–12, match the inequality with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]



7. $x < 3$
8. $x \geq 5$
9. $-3 < x \leq 4$
10. $0 \leq x \leq \frac{9}{2}$
11. $|x| < 3$
12. $|x| > 4$

In Exercises 13–18, determine whether each value of *x* is a solution of the inequality.

Inequality	*Values*	
13. $5x - 12 > 0$	(a) $x = 3$	(b) $x = -3$
	(c) $x = \frac{5}{2}$	(d) $x = \frac{3}{2}$
14. $2x + 1 < -3$	(a) $x = 0$	(b) $x = -\frac{1}{4}$
	(c) $x = -4$	(d) $x = -\frac{3}{2}$
15. $0 < \frac{x-2}{4} < 2$	(a) $x = 4$	(b) $x = 10$
	(c) $x = 0$	(d) $x = \frac{7}{2}$
16. $-1 < \frac{3-x}{2} \leq 1$	(a) $x = 0$	(b) $x = -5$
	(c) $x = 1$	(d) $x = 5$
17. $|x - 10| \geq 3$	(a) $x = 13$	(b) $x = -1$
	(c) $x = 14$	(d) $x = 9$
18. $|2x - 3| < 15$	(a) $x = -6$	(b) $x = 0$
	(c) $x = 12$	(d) $x = 7$

In Exercises 19–44, solve the inequality and sketch the solution on the real number line. (Some equalities have no solutions.)

19. $4x < 12$
20. $10x < -40$
21. $-2x > -3$
22. $-6x > 15$
23. $x - 5 \geq 7$
24. $x + 7 \leq 12$
25. $2x + 7 < 3 + 4x$
26. $3x + 1 \geq 2 + x$
27. $2x - 1 \geq 1 - 5x$
28. $6x - 4 \leq 2 + 8x$
29. $4 - 2x < 3(3 - x)$
30. $4(x + 1) < 2x + 3$
31. $\frac{3}{4}x - 6 \leq x - 7$
32. $3 + \frac{2}{7}x > x - 2$
33. $\frac{1}{2}(8x + 1) \geq 3x + \frac{5}{2}$
34. $9x - 1 < \frac{3}{4}(16x - 2)$
35. $3.6x + 11 \geq -3.4$
36. $15.6 - 1.3x < -5.2$
37. $1 < 2x + 3 < 9$
38. $-8 \leq -(3x + 5) < 13$
39. $-4 < \frac{2x - 3}{3} < 4$
40. $0 \leq \frac{x + 3}{2} < 5$
41. $\frac{3}{4} > x + 1 > \frac{1}{4}$
42. $-1 < 2 - \frac{x}{3} < 1$
43. $3.2 \leq 0.4x - 1 \leq 4.4$
44. $4.5 > \frac{1.5x + 6}{2} > 10.5$

In Exercises 45–60, solve the inequality and sketch the solution on the real number line. (Some inequalities have no solution.)

45. $|x| < 6$
46. $|x| > 4$
47. $\left|\frac{x}{2}\right| > 1$
48. $\left|\frac{x}{5}\right| > 3$
49. $|x - 5| < -1$
50. $|x - 7| < -5$
51. $|x - 20| \leq 6$
52. $|x - 8| \geq 0$
53. $|3 - 4x| \geq 9$
54. $|1 - 2x| < 5$
55. $\left|\frac{x - 3}{2}\right| \geq 4$
56. $\left|1 - \frac{2x}{3}\right| < 1$
57. $|9 - 2x| - 2 < -1$
58. $|x + 14| + 3 > 17$
59. $2|x + 10| \geq 9$
60. $3|4 - 5x| \leq 9$

Graphical Analysis **In Exercises 61–68, use a graphing utility to graph the inequality and identify the solution set.**

61. $6x > 12$

62. $3x - 1 \leq 5$

63. $5 - 2x \geq 1$

64. $3(x + 1) < x + 7$

65. $|x - 8| \leq 14$

66. $|2x + 9| > 13$

67. $2|x + 7| \geq 13$

68. $\frac{1}{2}|x + 1| \leq 3$

Graphical Analysis **In Exercises 69–74, use a graphing utility to graph the equation. Use the graph to approximate the values of *x* that satisfy each inequality.**

	Equation	*Inequalities*	
69.	$y = 2x - 3$	(a) $y \geq 1$	(b) $y \leq 0$
70.	$y = \frac{2}{3}x + 1$	(a) $y \leq 5$	(b) $y \geq 0$
71.	$y = -\frac{1}{2}x + 2$	(a) $0 \leq y \leq 3$	(b) $y \geq 0$
72.	$y = -3x + 8$	(a) $-1 \leq y \leq 3$	(b) $y \leq 0$
73.	$y = \|x - 3\|$	(a) $y \leq 2$	(b) $y \geq 4$
74.	$y = \left\|\frac{1}{2}x + 1\right\|$	(a) $y \leq 4$	(b) $y \geq 1$

In Exercises 75–80, find the interval(s) on the real number line for which the radicand is nonnegative (greater than or equal to zero).

75. $\sqrt{x - 5}$

76. $\sqrt{x - 10}$

77. $\sqrt{x + 3}$

78. $\sqrt{3 - x}$

79. $\sqrt[4]{7 - 2x}$

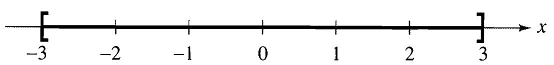
80. $\sqrt[4]{6x + 15}$

81. ***Think About It*** The graph of $|x - 5| < 3$ can be described as all real numbers within 3 units of 5. Give a similar description of $|x - 10| < 8$.

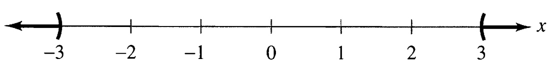
82. ***Think About It*** The graph of $|x - 2| > 5$ can be described as all real numbers more than 5 units from 2. Give a similar description of $|x - 8| > 4$.

In Exercises 83–90, use absolute value notation to define the interval (or pair of intervals) on the real number line.

83.



84.



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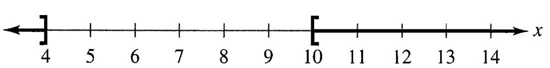

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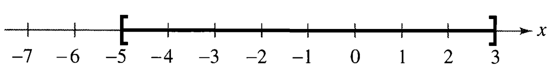


87. All real numbers within 10 units of 12

88. All real numbers at least 5 units from 8

89. All real numbers more than 5 units from -3

90. All real numbers no more than 7 units from -6

91. ***Car Rental*** You can rent a midsize car from Company A for \$250 per week with unlimited mileage. A similar car can be rented from Company B for \$150 per week plus 25 cents for each mile driven. How many miles must you drive in a week in order for the rental fee for Company B to be greater than that for Company A?

92. ***Copying Costs*** Your department sends its copying to the photocopy center of your company. The center bills your department \$0.10 per page. You have investigated the possibility of buying a departmental copier for \$3000. With your own copier, the cost per page would be \$0.03. The expected life of the copier is 4 years. How many copies must you make in the four-year period to justify buying the copier?

93. ***Investment*** In order for an investment of \$1000 to grow to more than \$1062.50 in 2 years, what must the annual interest rate be? $[A = P(1 + rt)]$

94. ***Investment*** In order for an investment of \$750 to grow to more than \$825 in 2 years, what must the annual interest rate be? $[A = P(1 + rt)]$

95. ***Cost, Revenue, and Profit*** The revenue for selling x units of a product is $R = 115.95x$. The cost of producing x units is

$$C = 95x + 750.$$

To obtain a profit, the revenue must be greater than the cost. For what values of x will this product return a profit?

96. ***Cost, Revenue, and Profit*** The revenue for selling x units of a product is $R = 24.55x$. The cost of producing x units is

$$C = 15.4x + 150{,}000.$$

To obtain a profit, the revenue must be greater than the cost. For what values of x will this product return a profit?

97. ***Data Analysis*** The admissions office of a college wants to determine whether there is a relationship between IQ scores x and grade-point averages y after the first year of school. An equation that models the data the admissions office obtained is

$y = 0.067x - 5.638.$

(a) Use a graphing utility to graph the model.

(b) Use the graph to estimate the values of x that predict a grade-point average of at least 3.0.

▶ Model It

98. ***Data Analysis*** You want to determine whether there is a relationship between an athlete's weight x (in pounds) and the athlete's maximum bench-press weight y (in pounds). The table shows a sample of data from 12 athletes.

Athlete's weight, x	Bench-press weight, y
165	170
184	185
150	200
210	255
196	205
240	295
202	190
170	175
185	195
190	185
230	250
160	155

(a) Use a graphing utility to plot the data.

(b) A model for this data is $y = 1.3x - 36$. Use a graphing utility to graph the model in the same viewing window used in part (a).

(c) Use the graph to estimate the values of x that predict a maximum bench-press weight of at least 200 pounds.

(d) Verify the estimate from part (c) algebraically.

▶ Model It *(continued)*

(e) Use the graph to write a statement about the accuracy of the model. If you think the graph indicates that an athlete's weight is not a particularly good indicator of the athlete's maximum bench-press weight, list other factors that might influence an individual's maximum bench-press weight.

99. ***Teachers' Salaries*** The average salary S (in thousands of dollars) for elementary and secondary teachers in the United States from 1980 to 2000 is approximated by the model

$S = 1.33t + 16.8, \quad 0 \le t \le 19$

where $t = 0$ represents 1980. According to this model, when will the average teacher's salary exceed \$45,000? (Source: National Education Association)

100. ***Egg Production*** The number of eggs E (in billions) produced in the United States from 1990 to 1999 can be modeled by $E = 1.55t + 67.5$, where $t = 0$ represents 1990. According to the model, when will the number of eggs produced exceed 88 billion? (Source: U.S. Department of Agriculture)

101. ***Geometry*** The side of a square is measured as 10.4 inches with a possible error of $\frac{1}{16}$ inch. Using these measurements, determine the interval containing the possible areas of the square.

102. ***Geometry*** The side of a square is measured as 24.2 centimeters with a possible error of 0.25 centimeter. Using these measurements, determine the interval containing the possible areas of the square.

103. ***Accuracy of Measurement*** You buy a bag of oranges for \$0.95 per pound. The weight that is listed on the bag is 4.65 pounds. The scale that weighed the bag is accurate to within 1 ounce. How much might you have been undercharged or overcharged?

104. ***Accuracy of Measurement*** You buy six T-bone steaks that cost \$3.98 per pound. The weight that is listed on the package is 5.72 pounds. The scale that weighed the package is accurate to within $\frac{1}{2}$ ounce. How much might you have been undercharged or overcharged?

105. ***Time Study*** A time study was conducted to determine the length of time required to perform a particular task in a manufacturing process. The times required by approximately two-thirds of the workers in the study satisfied the inequality

$$\left|\frac{t - 15.6}{1.9}\right| < 1$$

where t is time in minutes. Determine the interval on the real number line in which these times lie.

106. ***Height*** The heights h of two-thirds of the members of a population satisfy the inequality

$$\left|\frac{h - 68.5}{2.7}\right| \leq 1$$

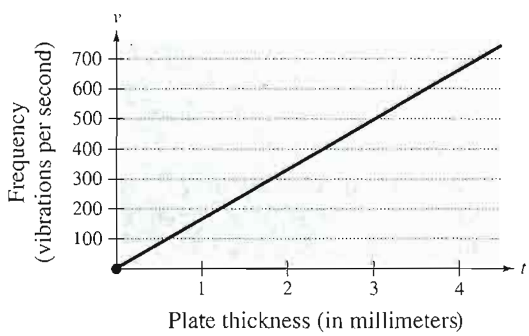
where h is measured in inches. Determine the interval on the real number line in which these heights lie.

107. ***Meteorology*** An electronic device is to be operated in an environment with relative humidity h in the interval defined by $|h - 50| \leq 30$. What are the minimum and maximum relative humidities for the operation of this device?

108. ***Music*** Michael Kasha of Florida State University used physics and mathematics to design a new classical guitar. He used the model for the frequency of the vibrations on a circular plate

$$v = \frac{2.6t}{d^2}\sqrt{\frac{E}{\rho}}$$

where v is the frequency (in vibrations per second), t is the plate thickness (in millimeters), d is the diameter of the plate, E is the elasticity of the plate material, and ρ is the density of the plate material. For fixed values of d, E, and ρ, the graph of the equation is a line (see figure).



(a) Estimate the frequency when the plate thickness is 2 millimeters.

(b) Estimate the plate thickness when the frequency is 600 vibrations per second.

(c) Approximate the interval for the plate thickness when the frequency is between 200 and 400 vibrations per second.

(d) Approximate the interval for the frequency when the plate thickness is less than 3 millimeters.

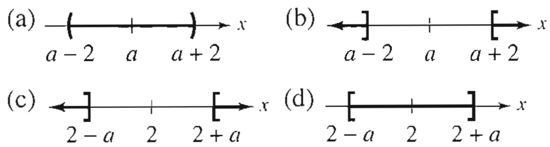
Synthesis

True or False? **In Exercises 109 and 110, determine whether the statement is true or false. Justify your answer.**

109. If a, b, and c are real numbers, and $a \leq b$, then $ac \leq bc$.

110. If $-10 \leq x \leq 8$, then $-10 \geq -x$ and $-x \geq -8$.

111. Identify the graph of the inequality $|x - a| \geq 2$.



112. Find sets of values of a, b, and c such that $0 \leq x \leq 10$ is a solution of the inequality $|ax - b| \leq c$.

Review

In Exercises 113–116, find the distance between each pair of points. Then find the midpoint of the line segment joining the points.

113. $(-4, 2), (1, 12)$ **114.** $(1, -2), (10, 3)$

115. $(3, 6), (-5, -8)$ **116.** $(0, -3), (-6, 9)$

In Exercises 117–124, solve the equation.

117. $3(x - 1) = 30$ **118.** $8x - 5(x + 4) = -19$

119. $-6(2 - x) - 12 = 36$

120. $4(x + 7) - 9 = -6(-x - 1)$

121. $2x^2 - 19x - 10 = 0$ **122.** $3x^2 - x - 10 = 0$

123. $14x^2 + 5x - 1 = 0$

124. $x^3 + 5x^2 - 4x - 20 = 0$

125. Find the coordinates of the point located 3 units to the left of the y-axis and 10 units above the x-axis.

126. Determine the quadrant(s) in which the point (x, y) could be located if $y > 0$.

1.8 Other Types of Inequalities

▶ **What you should learn**

- How to solve polynomial inequalities
- How to solve rational inequalities
- How to use inequalities to model and solve real-life problems

▶ **Why you should learn it**

Inequalities can be used to model and solve real-life problems. For instance, in Exercise 71 on page 159, a polynomial inequality is used to model the percent of households owning a television and having cable in the United States.

Stephen Ferry/Getty Images

Polynomial Inequalities

To solve a polynomial inequality such as $x^2 - 2x - 3 < 0$, you can use the fact that a polynomial can change signs only at its zeros (the x-values that make the polynomial equal to zero). Between two consecutive zeros, a polynomial must be entirely positive or entirely negative. This means that when the real zeros of a polynomial are put in order, they divide the real number line into intervals in which the polynomial has no sign changes. These zeros are the **critical numbers** of the inequality, and the resulting intervals are the **test intervals** for the inequality. For instance, the polynomial above factors as

$$x^2 - 2x - 3 = (x + 1)(x - 3)$$

and has two zeros, $x = -1$ and $x = 3$. These zeros divide the real number line into three test intervals:

$$(-\infty, -1),\quad (-1, 3),\quad \text{and}\quad (3, \infty). \qquad \text{(See Figure 1.29.)}$$

So, to solve the inequality $x^2 - 2x - 3 < 0$, you need only test one value from each of these test intervals to determine whether the value satisfies the original inequality. If so, you can conclude that the interval is a solution of the inequality.

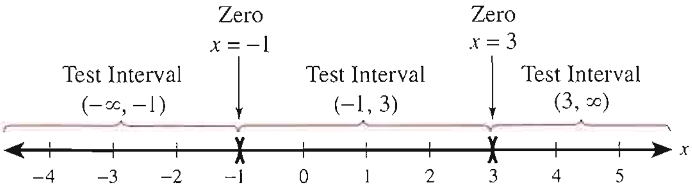


FIGURE 1.29 *Three test intervals for* $x^2 - 2x - 3$

You can use the same basic approach to determine the test intervals for any polynomial.

Finding Test Intervals for a Polynomial

To determine the intervals on which the values of a polynomial are entirely negative or entirely positive, use the following steps.

1. Find all real zeros of the polynomial, and arrange the zeros in increasing order (from smallest to largest). These zeros are the critical numbers of the polynomial.
2. Use the critical numbers of the polynomial to determine its test intervals.
3. Choose one representative x-value in each test interval and evaluate the polynomial at that value. If the value of the polynomial is negative, the polynomial will have negative values for every x-value in the interval. If the value of the polynomial is positive, the polynomial will have positive values for every x-value in the interval.

Example 1 ▶ Solving a Polynomial Inequality

Solve

$$x^2 - x - 6 < 0.$$

Solution

By factoring the polynomial as

$$x^2 - x - 6 = (x + 2)(x - 3)$$

you can see that the critical numbers are $x = -2$ and $x = 3$. So, the polynomial's test intervals are

$$(-\infty, -2), \quad (-2, 3), \quad \text{and} \quad (3, \infty). \qquad \text{Test intervals}$$

In each test interval, choose a representative x-value and evaluate the polynomial.

Interval	*x-Value*	*Polynomial Value*	*Conclusion*
$(-\infty, -2)$	$x = -3$	$(-3)^2 - (-3) - 6 = 6$	Positive
$(-2, 3)$	$x = 0$	$(0)^2 - (0) - 6 = -6$	Negative
$(3, \infty)$	$x = 4$	$(4)^2 - (4) - 6 = 6$	Positive

From this you can conclude that the inequality is satisfied for all x-values in $(-2, 3)$. This implies that the solution of the inequality $x^2 - x - 6 < 0$ is the interval $(-2, 3)$, as shown in Figure 1.30.

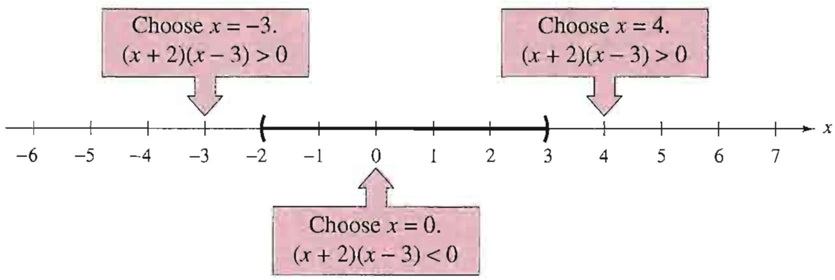


FIGURE 1.30

As with linear inequalities, you can check the reasonableness of a solution by substituting x-values into the original inequality. For instance, to check the solution found in Example 1, try substituting several x-values from the interval $(-2, 3)$ into the inequality

$$x^2 - x - 6 < 0.$$

Regardless of which x-values you choose, the inequality should be satisfied.

You can also use a graph to check the result of Example 1. Sketch the graph of $y = x^2 - x - 6$, as shown in Figure 1.31. Notice that the graph is below the x-axis on the interval $(-2, 3)$.

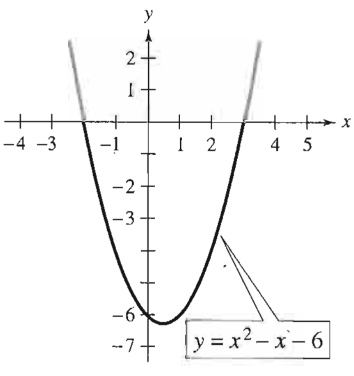


FIGURE 1.31

In Example 1, the polynomial inequality was given in general form (with the polynomial on one side and zero on the other). Whenever this is not the case, you should begin the solution process by writing the inequality in general form.

Example 2 ▶ Solving a Polynomial Inequality

Solve $2x^3 - 3x^2 - 32x > -48$

Solution

Begin by writing the inequality in general form.

$$2x^3 - 3x^2 - 32x > -48 \qquad \text{Write original inequality.}$$

$$2x^3 - 3x^2 - 32x + 48 > 0 \qquad \text{Write in general form.}$$

$$(x - 4)(x + 4)(2x - 3) > 0 \qquad \text{Factor.}$$

The critical numbers are $x = -4$, $x = \frac{3}{2}$, and $x = 4$, and the test intervals are $(-\infty, -4)$, $\left(-4, \frac{3}{2}\right)$, $\left(\frac{3}{2}, 4\right)$, and $(4, \infty)$.

Interval	*x-Value*	*Polynomial Value*	*Conclusion*
$(-\infty, -4)$	$x = -5$	$2(-5)^3 - 3(-5)^2 - 32(-5) + 48$	Negative
$\left(-4, \frac{3}{2}\right)$	$x = 0$	$2(0)^3 - 3(0)^2 - 32(0) + 48$	Positive
$\left(\frac{3}{2}, 4\right)$	$x = 2$	$2(2)^3 - 3(2)^2 - 32(2) + 48$	Negative
$(4, \infty)$	$x = 5$	$2(5)^3 - 3(5)^2 - 32(5) + 48$	Positive

From this you can conclude that the inequality is satisfied on the open intervals $\left(-4, \frac{3}{2}\right)$ and $(4, \infty)$. Therefore, the solution set consists of all real numbers in the intervals $\left(-4, \frac{3}{2}\right)$ and $(4, \infty)$, as shown in Figure 1.32.

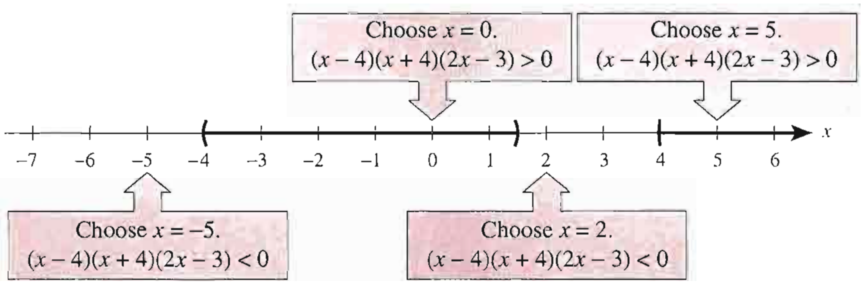


FIGURE 1.32

STUDY TIP

You may find it easier to determine the sign of a polynomial from its *factored* form. For instance, in Example 2, if the test value $x = 2$ is substituted into the factored form

$$(x - 4)(x + 4)(2x - 3)$$

you can see that the sign pattern of the factors is

$$(-)(+)(+)$$

which yields a negative result. Try using the factored form of the polynomial to determine the sign of the polynomial in the test intervals of the other examples in this section.

When solving a polynomial inequality, be sure you have accounted for the particular type of inequality symbol given in the inequality. For instance, in Example 2, note that the original inequality contained a "greater than" symbol and the solution consisted of two open intervals. If the original inequality had been

$$2x^3 - 3x^2 - 32x \geq 48$$

the solution would have consisted of the closed interval $\left[-4, \frac{3}{2}\right]$ and the interval $[4, \infty)$.

Each of the polynomial inequalities in Examples 1 and 2 has a solution set that consists of a single interval or the union of two intervals. When solving the exercises for this section, watch for unusual solution sets, as illustrated in Example 3.

Example 3 ▶ Unusual Solution Sets

a. The solution set of the following inequality consists of the entire set of real numbers, $(-\infty, \infty)$.

$$x^2 + 2x + 4 > 0$$

b. The solution set of the following inequality consists of the single real number $\{-1\}$.

$$x^2 + 2x + 1 \le 0$$

c. The solution set of the following inequality is empty.

$$x^2 + 3x + 5 < 0$$

d. The solution set of the following inequality consists of all real numbers except $x = 2$.

$$x^2 - 4x + 4 > 0$$

Exploration

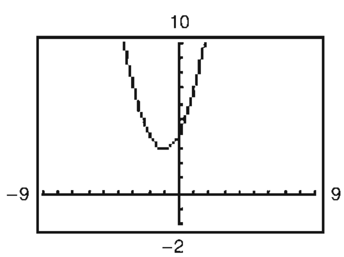
You can use a graphing utility to verify the results in Example 3. For instance, the graph of

$$y = x^2 + 2x + 4$$

is shown below. Notice that the y-values are greater than 0 for all values of x, as stated in Example 3(a). Use the graphing utility to graph the following:

$$y = x^2 + 2x + 1 \qquad y = x^2 + 3x + 5 \qquad y = x^2 - 4x + 4$$

Explain how you can use the graphs to verify the results of parts (b), (c), and (d) of Example 3.



Rational Inequalities

The concepts of critical numbers and test intervals can be extended to rational inequalities. To do this, use the fact that the value of a rational expression can change sign only at its *zeros* (the x-values for which its numerator is zero) and its *undefined values* (the x-values for which its denominator is zero). These two types of numbers make up the *critical numbers* of a rational inequality.

Example 4 ▶ Solving a Rational Inequality

Solve $\dfrac{2x-7}{x-5} \le 3$.

Solution

Begin by writing the rational inequality in general form.

$$\frac{2x-7}{x-5} \le 3 \qquad \text{Write original inequality.}$$

$$\frac{2x-7}{x-5} - 3 \le 0 \qquad \text{Write in general form.}$$

$$\frac{2x-7-3x+15}{x-5} \le 0 \qquad \text{Add fractions.}$$

$$\frac{-x+8}{x-5} \le 0 \qquad \text{Simplify.}$$

Critical numbers: $x = 5, x = 8$ Zeros and undefined values of rational expression

Test intervals: $(-\infty, 5), (5, 8), (8, \infty)$

Test: Is $\dfrac{-x+8}{x-5} \le 0$?

After testing these intervals, as shown in Figure 1.33, you can see that the inequality is satisfied on the open intervals $(-\infty, 5)$ and $(8, \infty)$. Moreover, because $(-x+8)/(x-5) = 0$ when $x = 8$, you can conclude that the solution set consists of all real numbers in the intervals $(-\infty, 5) \cup [8, \infty)$. (Be sure to use a closed interval to indicate that x can equal 8.)

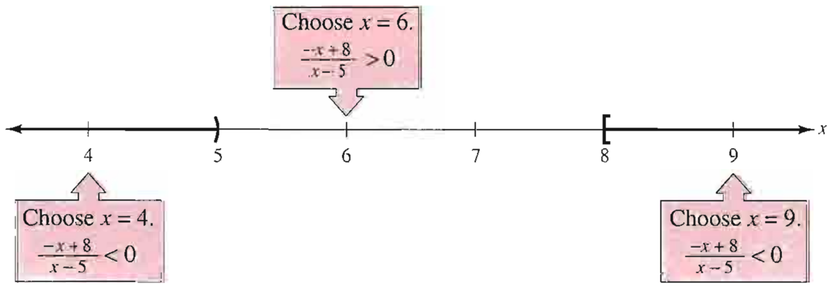


FIGURE 1.33

Applications

One common application of inequalities comes from business and involves profit, revenue, and cost. The formula that relates these three quantities is

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

$$P = R - C.$$

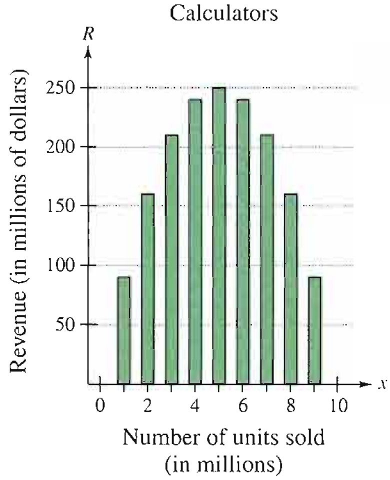


FIGURE 1.34

Example 5 ▶ Increasing the Profit for a Product

The marketing department of a calculator manufacturer has determined that the demand for a new model of calculator is

$$p = 100 - 0.00001x, \qquad 0 \le x \le 10{,}000{,}000 \qquad \text{Demand equation}$$

where p is the price per calculator (in dollars) and x represents the number of calculators sold. (If this model is accurate, no one would be willing to pay \$100 for the calculator. At the other extreme, the company couldn't sell more than 10 million calculators.) The revenue for selling x calculators is

$$R = xp = x(100 - 0.00001x) \qquad \text{Revenue equation}$$

as shown in Figure 1.34. The total cost of producing x calculators is \$10 per calculator plus a development cost of \$2,500,000. So, the total cost is

$$C = 10x + 2{,}500{,}000. \qquad \text{Cost equation}$$

What price should the company charge per calculator to obtain a profit of at least \$190,000,000?

Solution

Verbal Model: Profit = Revenue − Cost

Equation: $P = R - C$

$$P = 100x - 0.00001x^2 - (10x + 2{,}500{,}000)$$

$$P = -0.00001x^2 + 90x - 2{,}500{,}000$$

To answer the question, solve the inequality

$$P \ge 190{,}000{,}000$$

$$-0.00001x^2 + 90x - 2{,}500{,}000 \ge 190{,}000{,}000.$$

When you write the inequality in general form, find the critical numbers and the test intervals, and then test a value in each test interval, you can find the solution to be

$$3{,}500{,}000 \le x \le 5{,}500{,}000$$

as shown in Figure 1.35. Substituting the x-values in the original price equation shows that prices of

$$\$45.00 \le p \le \$65.00$$

will yield a profit of at least \$190,000,000.

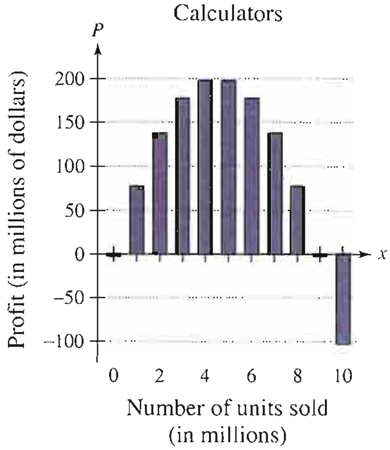


FIGURE 1.35

Another common application of inequalities is finding the domain of an expression that involves a square root, as shown in Example 6.

Example 6 ▶ Finding a Domain of an Expression

Find the domain of $\sqrt{64 - 4x^2}$.

Solution

Remember that the domain of an expression is the set of all x-values for which the expression is defined. Because $\sqrt{64 - 4x^2}$ is defined (has real values) only if $64 - 4x^2$ is nonnegative, the domain is given by $64 - 4x^2 \geq 0$.

$$64 - 4x^2 \geq 0 \qquad \text{Write in general form.}$$

$$16 - x^2 \geq 0 \qquad \text{Divide each side by 4.}$$

$$(4 - x)(4 + x) \geq 0 \qquad \text{Write in factored form.}$$

So, the inequality has two critical numbers: $x = -4$ and $x = 4$. You can use these two numbers to test the inequality as follows.

Critical numbers: $x = -4, x = 4$

Test intervals: $(-\infty, -4), (-4, 4), (4, \infty)$

Test: Is $(4 - x)(4 + x) \geq 0$?

A test shows that the inequality is satisfied in the *closed interval* $[-4, 4]$. So, the domain of the expression $\sqrt{64 - 4x^2}$ is the interval $[-4, 4]$, as shown in Figure 1.36.

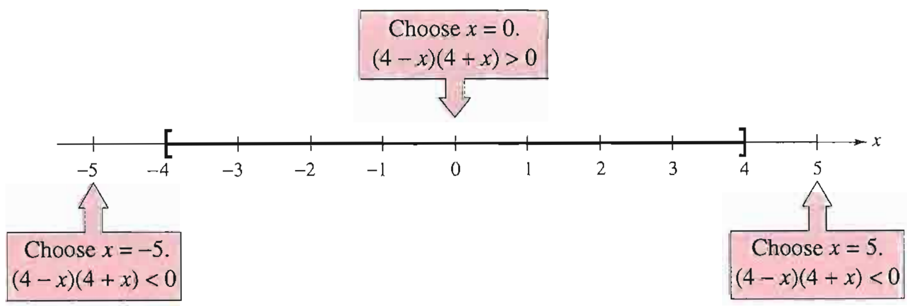


FIGURE 1.36

Writing ABOUT MATHEMATICS

Profit Analysis Consider the relationship $P = R - C$ described on page 156. Write a paragraph discussing why it might be beneficial to solve $P < 0$ if you owned a business. Use the situation described in Example 5 to illustrate your reasoning.

1.8 Exercises

In Exercises 1–4, determine whether each value of x is a solution of the inequality.

Inequality	*Values*	
1. $x^2 - 3 < 0$	(a) $x = 3$	(b) $x = 0$
	(c) $x = \frac{3}{2}$	(d) $x = -5$
2. $x^2 - x - 12 \geq 0$	(a) $x = 5$	(b) $x = 0$
	(c) $x = -4$	(d) $x = -3$
3. $\dfrac{x+2}{x-4} \geq 3$	(a) $x = 5$	(b) $x = 4$
	(c) $x = -\frac{9}{2}$	(d) $x = \frac{9}{2}$
4. $\dfrac{3x^2}{x^2+4} < 1$	(a) $x = -2$	(b) $x = -1$
	(c) $x = 0$	(d) $x = 3$

In Exercises 5–8, find the critical numbers.

5. $2x^2 - x - 6$

6. $9x^3 - 25x^2$

7. $2 + \dfrac{3}{x-5}$

8. $\dfrac{x}{x+2} - \dfrac{2}{x-1}$

In Exercises 9–24, solve the inequality and graph the solution on the real number line.

9. $x^2 \leq 9$

10. $x^2 < 36$

11. $(x + 2)^2 < 25$

12. $(x - 3)^2 \geq 1$

13. $x^2 + 4x + 4 \geq 9$

14. $x^2 - 6x + 9 < 16$

15. $x^2 + x < 6$

16. $x^2 + 2x > 3$

17. $x^2 + 2x - 3 < 0$

18. $x^2 - 4x - 1 > 0$

19. $x^2 + 8x - 5 \geq 0$

20. $-2x^2 + 6x + 15 \leq 0$

21. $x^3 - 3x^2 - x + 3 > 0$

22. $x^3 + 2x^2 - 4x - 8 \leq 0$

23. $x^3 - 2x^2 - 9x - 2 \geq -20$

24. $2x^3 + 13x^2 - 8x - 46 \geq 6$

In Exercises 25–30, solve the inequality and write the solution set in interval notation.

25. $4x^3 - 6x^2 < 0$

26. $4x^3 - 12x^2 > 0$

27. $x^3 - 4x \geq 0$

28. $2x^3 - x^4 \leq 0$

29. $(x - 1)^2(x + 2)^3 \geq 0$

30. $x^4(x - 3) \leq 0$

Graphical Analysis **In Exercises 31–34, use a graphing utility to graph the equation. Use the graph to approximate the values of x that satisfy each inequality.**

Equation	*Inequalities*	
31. $y = -x^2 + 2x + 3$	(a) $y \leq 0$	(b) $y \geq 3$
32. $y = \frac{1}{2}x^2 - 2x + 1$	(a) $y \leq 0$	(b) $y \geq 7$
33. $y = \frac{1}{8}x^3 - \frac{1}{2}x$	(a) $y \geq 0$	(b) $y \leq 6$
34. $y = x^3 - x^2 - 16x + 16$	(a) $y \leq 0$	(b) $y \geq 36$

In Exercises 35–48, solve the inequality and graph the solution on the real number line.

35. $\dfrac{1}{x} - x > 0$

36. $\dfrac{1}{x} - 4 < 0$

37. $\dfrac{x+6}{x+1} - 2 < 0$

38. $\dfrac{x+12}{x+2} - 3 \geq 0$

39. $\dfrac{3x-5}{x-5} > 4$

40. $\dfrac{5+7x}{1+2x} < 4$

41. $\dfrac{4}{x+5} > \dfrac{1}{2x+3}$

42. $\dfrac{5}{x-6} > \dfrac{3}{x+2}$

43. $\dfrac{1}{x-3} \leq \dfrac{9}{4x+3}$

44. $\dfrac{1}{x} \geq \dfrac{1}{x+3}$

45. $\dfrac{x^2+2x}{x^2-9} \leq 0$

46. $\dfrac{x^2+x-6}{x} \geq 0$

47. $\dfrac{5}{x-1} - \dfrac{2x}{x+1} < 1$

48. $\dfrac{3x}{x-1} \leq \dfrac{x}{x+4} + 3$

Graphical Analysis **In Exercises 49–52, use a graphing utility to graph the equation. Use the graph to approximate the values of x that satisfy each inequality.**

Equation	*Inequalities*	
49. $y = \dfrac{3x}{x-2}$	(a) $y \leq 0$	(b) $y \geq 6$
50. $y = \dfrac{2(x-2)}{x+1}$	(a) $y \leq 0$	(b) $y \geq 8$
51. $y = \dfrac{2x^2}{x^2+4}$	(a) $y \geq 1$	(b) $y \leq 2$
52. $y = \dfrac{5x}{x^2+4}$	(a) $y \geq 1$	(b) $y \leq 0$

In Exercises 53–58, find the domain of x in the expression.

53. $\sqrt{4 - x^2}$

54. $\sqrt{x^2 - 4}$

55. $\sqrt{x^2 - 7x + 12}$

56. $\sqrt{144 - 9x^2}$

57. $\sqrt{\dfrac{x}{x^2 - 2x - 35}}$

58. $\sqrt{\dfrac{x}{x^2 - 9}}$

In Exercises 59–64, solve the inequality. (Round your answers to two decimal places.)

59. $0.4x^2 + 5.26 < 10.2$

60. $-1.3x^2 + 3.78 > 2.12$

61. $-0.5x^2 + 12.5x + 1.6 > 0$

62. $1.2x^2 + 4.8x + 3.1 < 5.3$

63. $\dfrac{1}{2.3x - 5.2} > 3.4$

64. $\dfrac{2}{3.1x - 3.7} > 5.8$

65. ***Physics*** A projectile is fired straight upward from ground level with an initial velocity of 160 feet per second.

(a) At what instant will it be back at ground level?

(b) When will the height exceed 384 feet?

66. ***Physics*** A projectile is fired straight upward from ground level with an initial velocity of 128 feet per second.

(a) At what instant will it be back at ground level?

(b) When will the height be less than 128 feet?

67. ***Geometry*** A rectangular playing field with a perimeter of 100 meters is to have an area of at least 500 square meters. Within what bounds must the length of the rectangle lie?

68. ***Geometry*** A rectangular parking lot with a perimeter of 440 feet is to have an area of at least 8000 square feet. Within what bounds must the length of the rectangle lie?

69. ***Investment*** P dollars, invested at interest rate r compounded annually, increases to an amount

$$A = P(1 + r)^2$$

in 2 years. An investment of \$1000 is to increase to an amount greater than \$1100 in 2 years. The interest rate must be greater than what percent?

70. ***Cost, Revenue, and Profit*** The revenue and cost equations for a product are

$$R = x(50 - 0.0002x) \quad \text{and} \quad C = 12x + 150{,}000$$

where R and C are measured in dollars and x represents the number of units sold. How many units must be sold to obtain a profit of at least \$1,650,000?

▶ Model It

71. ***Cable Television*** The percent C of households in the United States that owned a television and had cable from 1970 to 2000 can be modeled by

$$C = 0.0002t^4 - 0.018t^3 + 0.48t^2 - 1.5t + 8$$

where t is the year, with $t = 0$ corresponding to 1970. (Source: Nielsen Media Research)

(a) Use a graphing utility to graph the equation.

(b) Complete the table to determine the year in which the percent of households that own a television and have cable will exceed 72%.

t	30	32	34	36	38	40
C						

(c) Use the *trace* feature of a graphing utility to verify your answer to part (b).

(d) Complete the table to determine the years during which the percent of households that own a television and have cable will be between 72% and 100%.

t	40	41	42	43	44	45	46	47
C								

(e) Use the *trace* feature of a graphing utility to verify your answer to part (d).

(f) Explain why the model may have values greater than 100% even though such values are not reasonable.

72. ***Safe Load*** The maximum safe load uniformly distributed over a one-foot section of a two-inch-wide wooden beam is approximated by the model

$$\text{Load} = 168.5d^2 - 472.1$$

where d is the depth of the beam.

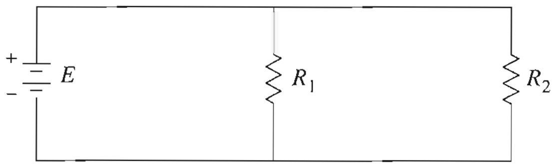
(a) Evaluate the model for $d = 4$, $d = 6$, $d = 8$, $d = 10$, and $d = 12$. Use the results to create a bar graph.

(b) Determine the minimum depth of the beam that will safely support a load of 2000 pounds.

73. ***Resistors*** When two resistors of resistance R_1 and R_2 are connected in parallel (see figure), the total resistance R satisfies the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

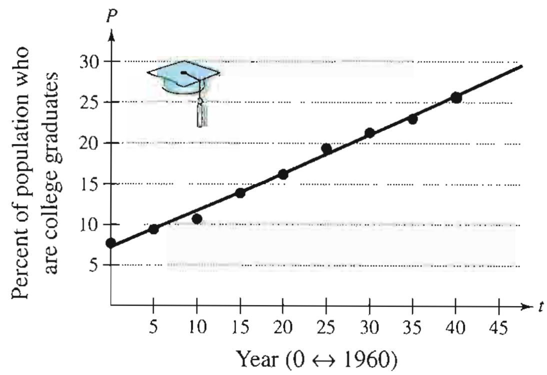
Find R_1 for a parallel circuit in which $R_2 = 2$ ohms and R must be at least 1 ohm.



74. ***Education*** The percent P of the U.S. population that had completed 4 years of college or more from 1960 to 1999 is approximated by the model

$$P = 0.0006t^2 + 0.441t + 7.21$$

where t is the year, with $t = 0$ corresponding to 1960. According to this model, during what year will more than 27% of the population be college graduates? (Source: U.S. Census Bureau)



Synthesis

True or False? **In Exercises 75 and 76, determine whether the statement is true or false. Justify your answer.**

75. The zeros of the polynomial $x^3 - 2x^2 - 11x + 12 \geq 0$ divide the real number line into four test intervals.

76. The solution set of the inequality $\frac{3}{2}x^2 + 3x + 6 \geq 0$ is the entire set of real numbers.

Exploration **In Exercises 77–80, find the interval for b such that the equation has at least one real solution.**

77. $x^2 + bx + 4 = 0$

78. $x^2 + bx - 4 = 0$

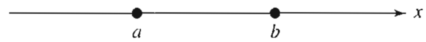
79. $3x^2 + bx + 10 = 0$

80. $2x^2 + bx + 5 = 0$

81. (a) Write a conjecture about the interval for b in Exercises 77–80. Explain your reasoning.

(b) What is the center of the interval for b in Exercises 77–80?

82. Consider the polynomial $(x - a)(x - b)$ and the real number line shown below.



(a) Identify the points on the line at which the polynomial is zero.

(b) In each of the three subintervals of the line, write the sign of each factor and the sign of the product.

(c) For what x-values does the polynomial change signs?

Review

In Exercises 83–86, factor the expression completely.

83. $4x^2 + 20x + 25$

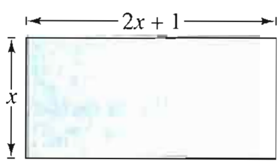
84. $(x + 3)^2 - 16$

85. $x^2(x + 3) - 4(x + 3)$

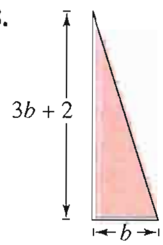
86. $2x^4 - 54x$

In Exercises 87 and 88, write an expression for the area of the region.

87.



88.



Chapter Summary

▶ *What* did you learn?

Section 1.1	Review Exercises
☐ How to sketch graphs of equations and find x- and y- intercepts of graphs of equations	1–12
☐ How to use symmetry to sketch graphs of equations	13–20
☐ How to find equations and sketch graphs of circles	21–28
☐ How to use graphs of equations in solving real-life problems	29, 30
Section 1.2	
☐ How to identify equations and solve linear equations in one variable	31–38
☐ How to solve equations that lead to linear equations	39–42
☐ How to find x- and y-intercepts of graphs of equations algebraically	43–50
☐ How to use linear equations to model and solve real-life problems	51, 52
Section 1.3	
☐ How to use a verbal model in a problem-solving plan	53, 54
☐ How to write and use mathematical models to solve real-life problems	55–58
☐ How to solve mixture problems and use common formulas to solve real-life problems	59–64
Section 1.4	
☐ How to solve quadratic equations by factoring, by extracting square roots, by completing the square, and by using the Quadratic Formula	65–74
☐ How to use quadratic equations to model and solve real-life problems	75, 76
Section 1.5	
☐ How to use the imaginary unit i to write complex numbers	77–80
☐ How to add, subtract, and multiply complex numbers	81–86
☐ How to use complex conjugates to write the quotient of two complex numbers in standard form	87–90
☐ How to find complex solutions of quadratic equations	91–94
Section 1.6	
☐ How to solve polynomial equations of degree three or greater and solve equations involving radicals, fractions, or absolute values	95–112
☐ How to use different types of equations to model and solve real-life problems	113, 114
Section 1.7	
☐ How to represent solutions of linear inequalities in one variable, solve linear inequalities in one variable, and solve inequalities involving absolute values	115–128
☐ How to use inequalities to model and solve real-life problems	129, 130
Section 1.8	
☐ How to solve polynomial and rational inequalities	131–138
☐ How to use inequalities to model and solve real-life problems	139, 140

Review Exercises

1.1 In Exercises 1–4, complete a table of values. Use the solution points to sketch the graph of the equation.

1. $y = 3x - 5$

2. $y = -\frac{1}{2}x + 2$

3. $y = x^2 - 3x$

4. $y = 2x^2 - x - 9$

In Exercises 5–10, sketch the graph *by hand*.

5. $y - 2x - 3 = 0$

6. $3x + 2y + 6 = 0$

7. $y = \sqrt{5 - x}$

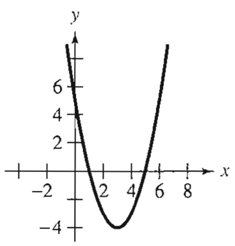
8. $y = \sqrt{x + 2}$

9. $y + 2x^2 = 0$

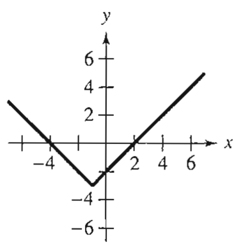
10. $y = x^2 - 4x$

In Exercises 11 and 12, find the *x*- and *y*-intercepts of the graph of the equation.

11. $y = (x - 3)^2 - 4$



12. $y = |x + 1| - 3$



In Exercises 13–20, use the algebraic tests to check for symmetry with respect to both axes and the origin. Then sketch the graph.

13. $y = -4x + 1$

14. $y = 5x - 6$

15. $y = 5 - x^2$

16. $y = x^2 - 10$

17. $y = x^3 + 3$

18. $y = -6 - x^3$

19. $y = \sqrt{x + 5}$

20. $y = |x| + 9$

In Exercises 21–26, find the center and radius of the circle and sketch its graph.

21. $x^2 + y^2 = 9$

22. $x^2 + y^2 = 4$

23. $(x + 2)^2 + y^2 = 16$

24. $x^2 + (y - 8)^2 = 81$

25. $\left(x - \frac{1}{2}\right)^2 + (y + 1)^2 = 36$

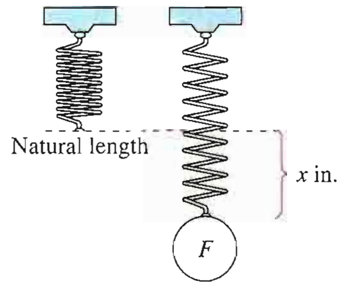
26. $(x + 4)^2 + \left(y - \frac{3}{2}\right)^2 = 100$

27. Find the standard form of the equation of the circle for which the endpoints of a diameter are $(0, 0)$ and $(4, -6)$.

28. Find the standard form of the equation of the circle for which the endpoints of a diameter are $(-2, -3)$ and $(4, -10)$.

29. ***Physics*** The force F (in pounds) required to stretch a spring x inches from its natural length (see figure) is

$$F = \frac{5}{4}x, \quad 0 \le x \le 20.$$



(a) Use the model to complete the table.

x	0	4	8	12	16	20
Force, F						

(b) Sketch a graph of the model.

(c) Use the graph to estimate the force necessary to stretch the spring 10 inches.

30. ***Number of Stores*** The number N of Home Depot stores from 1993 to 2000 can be approximated by the model

$$N = 9.53t^2 + 162$$

where t is the time (in years), with $t = 3$ corresponding to 1993. (Source: Home Depot, Inc.)

(a) Sketch a graph of the model.

(b) Use the graph to estimate the year in which the number of stores will be 2000.

1.2 In Exercises 31–34, determine whether the equation is an identity or a conditional equation.

31. $6 - (x - 2)^2 = 2 + 4x - x^2$

32. $3(x - 2) + 2x = 2(x + 3)$

33. $-x^3 + x(7 - x) + 3 = x(-x^2 - x) + 7(x + 1) - 4$

34. $3(x^2 - 4x + 8) = -10(x + 2) - 3x^2 + 6$

In Exercises 35–42, solve the equation (if possible) and check your solution.

35. $3x - 2(x + 5) = 10$

36. $4x + 2(7 - x) = 5$

37. $4(x + 3) - 3 = 2(4 - 3x) - 4$

38. $\frac{1}{2}(x - 3) - 2(x + 1) = 5$

39. $\dfrac{x}{5} - 3 = \dfrac{2x}{2} + 1$

40. $\dfrac{4x - 3}{6} + \dfrac{x}{4} = x - 2$

41. $\dfrac{18}{x} = \dfrac{10}{x - 4}$

42. $\dfrac{5}{x - 2} = \dfrac{13}{2x - 3}$

In Exercises 43–50, find the *x*- and *y*-intercepts of the graph of the equation algebraically.

43. $y = 3x - 1$

44. $y = -5x + 6$

45. $y = 2(x - 4)$

46. $y = 4(7x + 1)$

47. $y = -\frac{1}{2}x + \frac{2}{3}$

48. $y = \frac{3}{4}x - \frac{1}{4}$

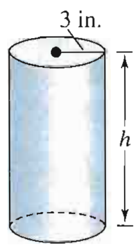
49. $3.8y - 0.5x + 1 = 0$

50. $1.5y + 2x - 1.2 = 0$

51. ***Geometry*** The surface area of the cylinder shown in the figure is approximated by

$$S = 2(3.14)(3)^2 + 2(3.14)(3)h.$$

The surface area is 244.92 square inches. Find the height h of the cylinder.



52. ***Temperature*** The Fahrenheit and Celsius temperature scales are related by the equation

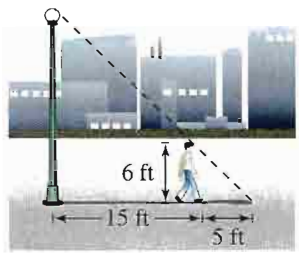
$$C = \frac{5}{9}F - \frac{160}{9}.$$

Find the Fahrenheit temperature that corresponds to 100° Celsius.

1.3 53. ***Profit*** In October, a greeting card company's total profit was 12% more than it was in September. The total profit for the two months was \$689,000. Write a verbal model, assign labels, and write an algebraic equation to find the profit for each month.

54. ***Discount*** The price of a television set has been discounted \$85. The sale price is \$340. Write a verbal model, assign labels, and write an algebraic equation to find the percent discount.

55. ***Shadow Length*** A person who is 6 feet tall walks away from a streetlight toward the tip of the streetlight's shadow. When the person is 15 feet from the streetlight, the tip of the person's shadow and the shadow cast by the streetlight coincide at a point 5 feet in front of the person (see figure). How tall is the streetlight?



56. ***Finance*** A group agrees to share equally in the cost of a \$48,000 piece of machinery. If it can find two more group members, each member's share will decrease by \$4000. How many are presently in the group?

57. ***Business Venture*** You are planning to start a small business that will require an investment of \$90,000. You have found some people who are willing to share equally in the venture. If you can find three more people, each person's share will decrease by \$2500. How many people have you found so far?

58. ***Average Speed*** You commute 56 miles one way to work. The trip to work takes 10 minutes longer than the trip home. Your average speed on the trip home is 8 miles per hour faster. What is your average speed on the trip home?

59. ***Mixture Problem*** A car radiator contains 10 liters of a 30% antifreeze solution. How many liters will have to be replaced with pure antifreeze if the resulting solution is to be 50% antifreeze?

60. ***Investment*** You invested \$6000 at $4\frac{1}{2}\%$ and $5\frac{1}{2}\%$ simple interest. During the first year, the two accounts earned \$305. How much did you invest in each?

In Exercises 61 and 62, solve for the indicated variable.

61. ***Volume of a Cone***

Solve for h: $V = \frac{1}{3}\pi r^2 h$

62. ***Kinetic Energy***

Solve for m: $E = \frac{1}{2}mv^2$

63. ***Travel Time*** Two cars start at a given time and travel in the same direction at average speeds of 40 miles per hour and 55 miles per hour. How much time will elapse before the two cars are 10 miles apart?

64. ***Geometry*** The volume of a circular cylinder is 81π cubic feet. The cylinder's radius is 3 feet. What is the height of the cylinder?

1.4 **In Exercises 65–74, use any method to solve the quadratic equation.**

65. $15 + x - 2x^2 = 0$

66. $2x^2 - x - 28 = 0$

67. $6 = 3x^2$

68. $16x^2 = 25$

69. $(x + 4)^2 = 18$

70. $(x - 8)^2 = 15$

71. $x^2 - 12x + 30 = 0$

72. $x^2 + 6x - 3 = 0$

73. $-2x^2 - 5x + 27 = 0$

74. $-20 - 3x + 3x^2 = 0$

75. ***Simply Supported Beam*** A simply supported 20-foot beam supports a uniformly distributed load of 1000 pounds per foot. The bending moment M (in foot-pounds) x feet from one end of the beam is given by $M = 500x(20 - x)$.

(a) Where is the bending moment zero?

(b) Use a graphing utility to graph the equation.

(c) Use the graph to determine the point on the beam where the bending moment is the greatest.

76. ***Physics*** A ball is thrown upward with an initial velocity of 30 feet per second from a point that is 24 feet above the ground. The height h (in feet) of the ball at time t (in seconds) after it is thrown is

$$h = -16t^2 + 30t + 24.$$

Find the time when the ball hits the ground.

1.5 **In Exercises 77–80, write the complex number in standard form.**

77. $6 + \sqrt{-4}$

78. $3 - \sqrt{-25}$

79. $i^2 + 3i$

80. $-5i + i^2$

In Exercises 81–86, perform the operation and write the result in standard form.

81. $(7 + 5i) + (-4 + 2i)$

82. $\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$

83. $5i(13 - 8i)$

84. $(1 + 6i)(5 - 2i)$

85. $(10 - 8i)(2 - 3i)$

86. $i(6 + i)(3 - 2i)$

In Exercises 87 and 88, write the quotient in standard form.

87. $\frac{6 + i}{4 - i}$

88. $\frac{3 + 2i}{5 + i}$

In Exercises 89 and 90, perform the operation and write the result in standard form.

89. $\frac{4}{2 - 3i} + \frac{2}{1 + i}$

90. $\frac{1}{2 + i} - \frac{5}{1 + 4i}$

In Exercises 91–94, find all solutions of the equation.

91. $3x^2 + 1 = 0$

92. $2 + 8x^2 = 0$

93. $x^2 - 2x + 10 = 0$

94. $6x^2 + 3x + 27 = 0$

1.6 **In Exercises 95–112, find all solutions of the equation. Check your solutions in the original equation.**

95. $5x^4 - 12x^3 = 0$

96. $4x^3 - 6x^2 = 0$

97. $x^4 - 5x^2 + 6 = 0$

98. $9x^4 + 27x^3 - 4x^2 - 12x = 0$

99. $\sqrt{x + 4} = 3$

100. $\sqrt{x - 2} - 8 = 0$

101. $\sqrt{2x + 3} + \sqrt{x - 2} = 2$

102. $5\sqrt{x} - \sqrt{x - 1} = 6$

103. $(x - 1)^{2/3} - 25 = 0$

104. $(x + 2)^{3/4} = 27$

105. $(x + 4)^{1/2} + 5x(x + 4)^{3/2} = 0$

106. $8x^2(x^2 - 4)^{1/3} + (x^2 - 4)^{4/3} = 0$

107. $\frac{5}{x} = 1 + \frac{3}{x + 2}$

108. $\frac{6}{x} + \frac{8}{x + 5} = 3$

109. $|x - 5| = 10$

110. $|2x + 3| = 7$

111. $|x^2 - 3| = 2x$

112. $|x^2 - 6| = x$

113. ***Demand*** The demand equation for a hair dryer is

$$p = 42 - \sqrt{0.001x + 2}$$

where x is the number of units demanded per day and p is the price per unit. Find the demand if the price is set at \$29.95.

114. ***Data Analysis*** The amount C of chlorofluorocarbon gases (CFCs) in thousands of metric tons emitted in the United States from 1993 to 1999 can be approximated by the model

$$C = 2.60t^2 - 48.7t + 269$$

where $t = 3$ represents 1993. The actual amounts emitted are shown in the table. (Source: U.S. Energy Information Administration)

Year, t	CFCs emitted, C
1993	148
1994	109
1995	102
1996	67
1997	51
1998	49
1999	41

(a) Use a graphing utility to compare the data with the model.

(b) Use the graph in part (a) to estimate the amount of CFCs emitted in 2005.

(c) Use the model to verify algebraically the estimate from part (b).

1.7 **In Exercises 115–118, write an inequality that represents the interval and state whether the interval is bounded or unbounded.**

115. $(-7, 2]$ **116.** $(4, \infty)$

117. $(-\infty, -10]$ **118.** $[-2, 2]$

In Exercises 119–128, solve the inequality.

119. $9x - 8 \le 7x + 16$ **120.** $\frac{15}{2}x + 4 > 3x - 5$

121. $4(5 - 2x) \le \frac{1}{2}(8 - x)$

122. $\frac{1}{2}(3 - x) > \frac{1}{3}(2 - 3x)$

123. $-19 < 3x - 17 \le 34$ **124.** $-3 \le \dfrac{2x - 5}{3} < 5$

125. $|x| \le 4$ **126.** $|x - 2| < 1$

127. $|x - 3| > 4$ **128.** $\left|x - \frac{3}{2}\right| \ge \frac{3}{2}$

129. ***Geometry*** The side of a square is measured as 19.3 centimeters with a possible error of 0.5 centimeter. Using these measurements, determine the interval containing the area of the square.

130. ***Cost, Revenue, and Profit*** The revenue for selling x units of a product is $R = 125.33x$. The cost of producing x units is $C = 92x + 1200$. To obtain a profit, the revenue must be greater than the cost. Determine the smallest value of x for which this product returns a profit.

1.8 **In Exercises 131–138, solve the inequality.**

131. $x^2 - 6x - 27 < 0$ **132.** $x^2 - 2x \ge 3$

133. $6x^2 + 5x < 4$ **134.** $2x^2 + x \ge 15$

135. $\dfrac{2}{x + 1} \le \dfrac{3}{x - 1}$ **136.** $\dfrac{x - 5}{3 - x} < 0$

137. $\dfrac{x^2 + 7x + 12}{x} \ge 0$ **138.** $\dfrac{1}{x - 2} > \dfrac{1}{x}$

139. ***Investment*** P dollars invested at interest rate r compounded annually increases to an amount $A = P(1 + r)^2$ in 2 years. An investment of \$5000 is to increase to an amount greater than \$5500 in 2 years. The interest rate must be greater than what percent?

140. ***Population of a Species*** A biologist introduces 200 ladybugs into a crop field. The population P of the ladybugs is approximated by the model

$$P = \frac{1000(1 + 3t)}{5 + t}$$

where t is the time in days. Find the time required for the population to increase to at least 2000 ladybugs.

Synthesis

True or False? **In Exercises 141 and 142, determine whether the statement is true or false. Justify your answer.**

141. $\sqrt{-18}\sqrt{-2} = \sqrt{(-18)(-2)}$

142. The equation $325x^2 - 717x + 398 = 0$ has no solution.

143. Explain why it is important to check your solutions to certain types of equations.

144. ***Error Analysis*** What is wrong with the following solution?

$$|11x + 4| \ge 26$$

$11x + 4 \le 26$ or $11x + 4 \ge 26$

$11x \le 22$ $\quad$ $11x \ge 22$

$x \le 2$ $\quad$ $x \ge 2$

Chapter Test

The *Interactive* CD-ROM and *Internet* versions of this text offer Chapter Pre-Tests and Chapter Post-Tests, both of which have randomly generated exercises with diagnostic capabilities.

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

In Exercises 1–6, check for symmetry with respect to both axes and the origin. Then sketch the graph of the equation. Identify any *x*- and *y*-intercepts.

1. $y = 4 - \frac{3}{4}x$

2. $y = 4 - \frac{3}{4}|x|$

3. $y = 4 - (x - 2)^2$

4. $y = x - x^3$

5. $y = \sqrt{3 - x}$

6. $(x - 3)^2 + y^2 = 9$

In Exercises 7–12, solve the equation (if possible).

7. $\frac{2}{3}(x - 1) + \frac{1}{4}x = 10$

8. $(x - 3)(x + 2) = 14$

9. $\dfrac{x - 2}{x + 2} + \dfrac{4}{x + 2} + 4 = 0$

10. $x^4 + x^2 - 6 = 0$

11. $2\sqrt{x} - \sqrt{2x + 1} = 1$

12. $|3x - 1| = 7$

In Exercises 13–16, solve the inequality. Sketch the solution on the real number line.

13. $-3 \le 2(x + 4) < 14$

14. $\dfrac{2}{x} > \dfrac{5}{x + 6}$

15. $2x^2 + 5x > 12$

16. $|x - 15| \ge 5$

17. Perform each operation and write the result in standard form.

(a) $10i - (3 + \sqrt{-25})$ (b) $(2 + \sqrt{3}i)(2 - \sqrt{3}i)$

18. Write the quotient in standard form: $\dfrac{5}{2 + i}$.

19. The sales y (in billions of dollars) for Gateway, Inc. from 1993 to 2000 can be approximated by the model

$$y = 1.16t - 1.9 \qquad 3 \le t \le 10$$

where t is the time (in years), with $t = 3$ corresponding to 1993. (Source: Gateway, Inc.)

(a) Sketch a graph of the model.

(b) Assuming that the pattern continues, use the graph in part (a) to estimate the sales in 2004.

(c) Use the model to verify algebraically the estimate from part (b).

20. On the first part of a 350-kilometer trip, a salesperson travels 2 hours and 15 minutes at an average speed of 100 kilometers per hour. The salesperson needs to arrive at the destination in another hour and 20 minutes. Find the average speed required for the remainder of the trip.

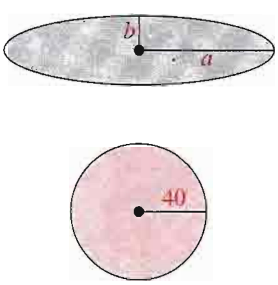


FIGURE FOR 21

21. The area of the ellipse in the figure at the left is $A = \pi ab$. If a and b satisfy the constraint $a + b = 100$, find a and b such that the area of the ellipse equals the area of the circle.

Proofs in Mathematics

Conditional Statements

Many theorems are written in the **if-then form** "if p, then q," which is denoted by

$p \to q$ Conditional statement

where p is the **hypothesis** and q is the **conclusion.** Here are some other ways to express the conditional statement $p \to q$.

p implies q. p, only if q. p is sufficient for q.

Conditional statements can be either true or false. The conditional statement $p \to q$ is false only when p is true and q is false. To show that a conditional statement is true, you must prove that the conclusion follows for all cases that fulfill the hypothesis. To show that a conditional statement is false, you need only to describe a single **counterexample** that shows that the statement is not always true.

For instance, $x = -4$ is a counterexample that shows that the following statement is false.

If $x^2 = 16$, then $x = 4$.

The hypothesis "$x^2 = 16$" is true because $(-4)^2 = 16$. However, the conclusion "$x = 4$" is false. This implies that the given conditional statement is false.

For the conditional statement $p \to q$, there are three important associated conditional statements.

1. The **converse** of $p \to q$: $q \to p$
2. The **inverse** of $p \to q$: $\sim p \to \sim q$
3. The **contrapositive** of $p \to q$: $\sim q \to \sim p$

The symbol $\sim$ means the **negation** of a statement. For instance, the negation of "The engine is running" is "The engine is not running."

Example ▶ Writing the Converse, Inverse, and Contrapositive

Write the converse, inverse, and contrapositive of the conditional statement "If I get a B on my test, then I will pass the course."

Solution

a. *Converse:* If I pass the course, then I got a B on my test.

b. *Inverse:* If I do not get a B on my test, then I will not pass the course.

c. *Contrapositive:* If I do not pass the course, then I did not get a B on my test.

In the example above, notice that neither the converse nor the inverse is logically equivalent to the original conditional statement. On the other hand, the contrapositive *is* logically equivalent to the original conditional statement.

P.S. Problem Solving

1. Let x represent the time (in seconds) and let y represent the distance (in feet) between you and a tree. Sketch a possible graph that shows how x and y are related if you are walking toward the tree.

2. (a) Find the following sums

$$1 + 2 + 3 + 4 + 5 =$$

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 =$$

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 =$$

(b) Use the following formula for the sum of the first n natural numbers to verify your answers to part (a).

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$$

(c) Use the formula in part (b) to find n if the sum of the first n natural numbers is 210.

3. The area of an ellipse is given by $A = \pi ab$ (see figure). For a certain ellipse, it is required that $a + b = 20$.

(a) Show that $A = \pi a(20 - a)$.

(b) Complete the table.

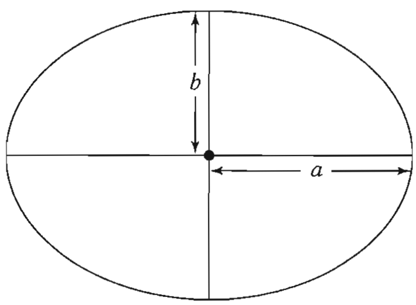
a	4	7	10	13	16
A					

(c) Find two values of a such that $A = 300$.

(d) Use a graphing utility to graph the area equation.

(e) Find the a-intercepts of the graph of the area equation. What do these values represent?

(f) What is the maximum area? What values of a and b yield the maximum area?



4. A building code requires that a building be able to withstand a certain amount of wind pressure. The pressure P (in pounds per square foot) from wind blowing at s miles per hour is given by

$$P = 0.00256s^2.$$

(a) A two-story library is designed. Buildings this tall are often required to withstand wind pressure of 20 pounds per square foot. Under this requirement, how fast can the wind be blowing before it produces excessive stress on the building?

(b) To be safe, the library is designed so that it can withstand wind pressure of 40 pounds per square foot. Does this mean that the library can survive wind blowing at twice the speed you found in part (a)? Justify your answer.

(c) Use the pressure formula to explain why even a relatively small increase in the wind speed could have potentially serious effects on a building.

5. For a bathtub with a rectangular base, Toricelli's Law implies that the height h of water in the tub t seconds after it begins draining is given by

$$h = \left(\sqrt{h_0} - \frac{2\pi d^2\sqrt{3}}{lw}t\right)^2$$

where l and w are the tub's length and width, d is the diameter of the drain, and h_0 is the water's initial height. (All measurements are in inches.) You completely fill a tub with water. The tub is 60 inches long by 30 inches wide by 25 inches high and has a drain with a two-inch diameter.

(a) Find the time it takes for the tub to go from being full to half-full.

(b) Find the time it takes for the tub to go from being half-full to empty.

(c) Based on your results in parts (a) and (b), what general statement can you make about the speed at which the water drains.

6. (a) Consider the sum of squares $x^2 + 9$. If the sum can be factored, then there are integers m and n such that $x^2 + 9 = (x + m)(x + n)$. Write two equations relating the sum and the product of m and n to the coefficients in $x^2 + 9$.

(b) Show that there are no integers m and n that satisfy both equations you wrote in part (a). What can you conclude?

7. A Pythagorean Triple is a group of three integers, such as 3, 4, and 5, that could be the lengths of the sides of a right triangle.

   (a) Find two other Pythagorean Triples.

   (b) Notice that $3 \cdot 4 \cdot 5 = 60$. Is the product of the three numbers in each Pythagorean Triple evenly divisible by 3? by 4? by 5?

   (c) Write a conjecture involving Pythagorean Triples and divisibility by 60.

8. Determine the solutions x_1 and x_2 of each quadratic equation. Use the values of x_1 and x_2 to fill in the boxes.

Equation	x_1, x_2	$x_1 + x_2$	$x_1 \cdot x_2$
(a) $x^2 - x - 6 = 0$			
(b) $2x^2 + 5x - 3 = 0$			
(c) $4x^2 - 9 = 0$			
(d) $x^2 - 10x + 34 = 0$			

9. Consider a general quadratic equation

   $ax^2 + bx + c = 0$

   whose solutions are x_1 and x_2. Use the results of Exercise 8 to determine a relationship among the coefficients a, b, and c and the sum $x_1 + x_2$ and the product $x_1 \cdot x_2$ of the solutions.

10. (a) The principal cube root of 125, $\sqrt[3]{125}$, is 5. Evaluate the expression x^3 for each value of x.

    (i) $x = \dfrac{-5 + 5\sqrt{3}i}{2}$

    (ii) $x = \dfrac{-5 - 5\sqrt{3}i}{2}$

    (b) The principal cube root of 27, $\sqrt[3]{27}$, is 3. Evaluate the expression x^3 for each value of x.

    (i) $x = \dfrac{-3 + 3\sqrt{3}i}{2}$

    (ii) $x = \dfrac{-3 - 3\sqrt{3}i}{2}$

    (c) Use the results of parts (a) and (b) to list possible cube roots of (i), 1, (ii) 8, and (iii) 64. Verify your results algebraically.

11. The multiplicative inverse of z is a complex number z_m such that $z \cdot z_m = 1$. Find the multiplicative inverse of each complex number.

    (a) $z = 1 + i$ (b) $z = 3 - i$ (c) $z = -2 + 8i$

12. Prove that the product of a complex number $a + bi$ and its complex conjugate is a real number.

13. A **fractal** is a geometric figure that consists of a pattern that is repeated infinitely on a smaller and smaller scale. The most famous fractal is called the **Mandelbrot Set**, named after the Polish-born mathematician Benoit Mandelbrot. To draw the Mandelbrot Set, consider the following sequence of numbers.

    $c, c^2 + c, (c^2 + c)^2 + c, [(c^2 + c)^2 + c]^2 + c, \ldots$

    The behavior of this sequence depends on the value of the complex number c. If the sequence is bounded (the absolute value of each number in the sequence, $|a + bi| = \sqrt{a^2 + b^2}$, is less than some fixed number N), the complex number c is in the Mandelbrot Set, and if the sequence is unbounded (the absolute value of the terms of the sequence become infinitely large), the complex number c is not in the Mandelbrot Set. Determine whether the complex number c is in the Mandelbrot Set.

    (a) $c = i$ (b) $c = 1 + i$ (c) $c = -2$

14. Use the equation

    $4\sqrt{x} = 2x + k$

    to find three different values of k such that the equation has two solutions, one solution, and no solution. Describe the process you used to find the equations.

15. Use the graph of $y = x^4 - x^3 - 6x^2 + 4x + 8$ to solve the inequality $x^4 - x^3 - 6x^2 + 4x + 8 > 0$.

16. When you buy a 16-ounce bag of chips, you expect to get precisely 16 ounces. The actual weight w (in ounces) of a "16-ounce" bag of chips is given by

    $|w - 16| \le \dfrac{1}{2}$.

    You buy four 16-ounce bags, what is the greatest amount you can expect to get? What is the smallest amount? Explain.