How to study Chapter 2

What you should learn

In this chapter you will learn the following skills and concepts:

- How to find and use the slopes of lines to write and graph linear equations in two variables
- How to evaluate functions and find their domains
- How to analyze graphs of functions
- How to identify and graph rigid and nonrigid transformations
- How to find arithmetic combinations and compositions of functions
- How to find inverse functions graphically and algebraically

Important Vocabulary

As you encounter each new vocabulary term in this chapter, add the term and its definition to your notebook glossary.

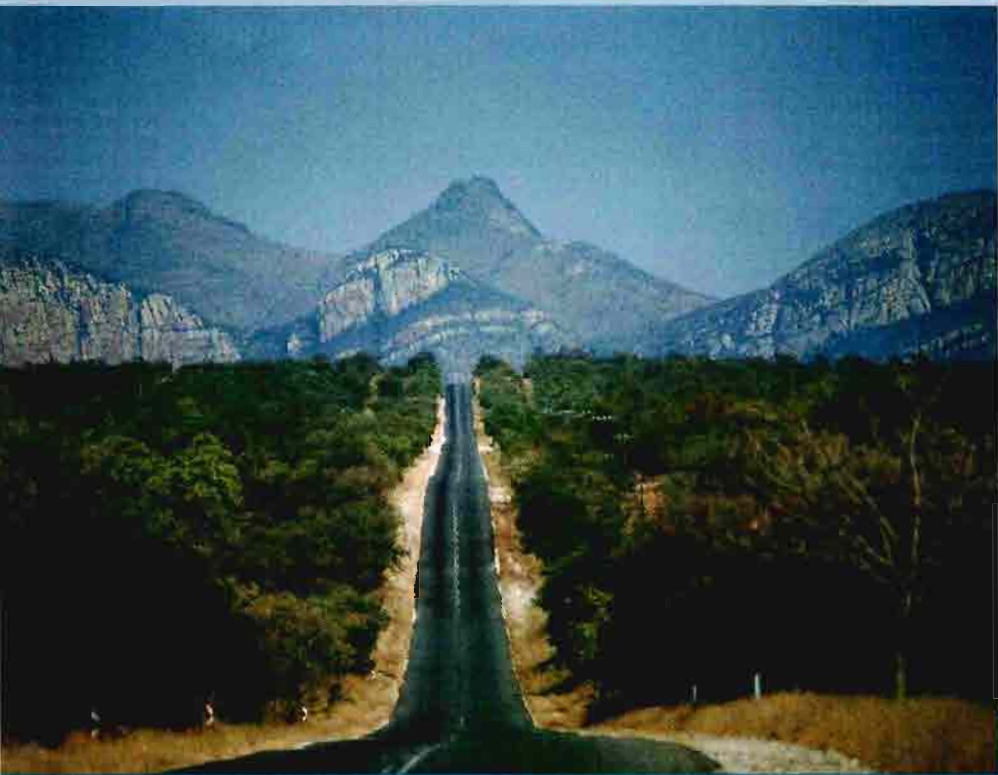
Linear equation in two variables (p. 172)
Slope (p. 172)
Slope-intercept form (p. 172)
Point-slope form (p. 177)
Parallel (p. 179)
Perpendicular (p. 179)
Function (p. 187)
Domain (p. 187)
Range (p. 187)
Independent variable (p. 189)
Dependent variable (p. 189)
Implied domain (p. 191)
Graph of a function (p. 201)
Vertical Line Test (p. 202)
Zeros of a function (p. 203)
Increasing function (p. 204)
Decreasing function (p. 204)
Constant function (p. 204)
Relative minimum (p. 205)
Relative maximum (p. 205)
Even function (p. 206)
Odd function (p. 206)
Vertical and horizontal shifts (p. 219)
Reflection (p. 221)
Nonrigid transformations (p. 223)
Inverse function (p. 237)
Horizontal Line Test (p. 240)
One-to-one functions (p. 240)

Study Tools

Learning objectives in each section
Chapter Summary (p. 247)
Review Exercises (pp. 248–251)
Chapter Test (p. 252)
Cumulative Test for Chapters P–2 (pp. 253–254)

Additional Resources

Study and Solutions Guide
Interactive College Algebra
Videotapes/DVD for Chapter 2
College Algebra Website
Student Success Organizer



Andreas Stirnberg/Getty Images

2 Functions and Their Graphs

2.1 Linear Equations in Two Variables

▶ What you should learn

- How to use slope to graph linear equations in two variables
- How to find slopes of lines
- How to write linear equations in two variables
- How to use slope to identify parallel and perpendicular lines
- How to use linear equations in two variables to model and solve real-life problems

▶ Why you should learn it

Linear equations in two variables can be used to model and solve real-life problems. For instance, in Exercise 119 on page 185, a linear equation is used to model the average monthly cellular phone bills for subscribers in the United States.

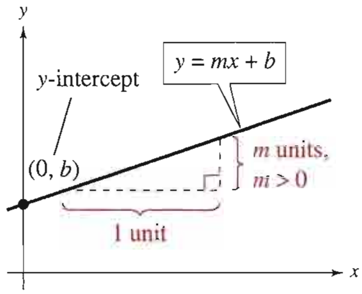
Dick Luria/Getty Images

Using Slope

The simplest mathematical model for relating two variables is the **linear equation in two variables** $y = mx + b$. The equation is called *linear* because its graph is a line. (In mathematics, the term *line* means *straight line*.) By letting $x = 0$, you can see that the line crosses the y-axis at $y = b$, as shown in Figure 2.1. In other words, the y-intercept is $(0, b)$. The steepness or slope of the line is m.

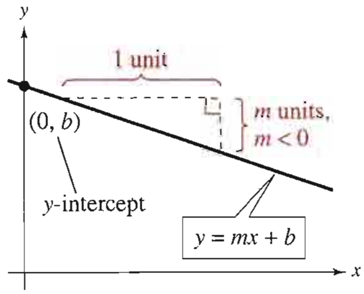
$$y = \underbrace{m}_{\text{Slope}}x + \underbrace{b}_{y\text{-Intercept}}$$

The **slope** of a nonvertical line is the number of units the line rises (or falls) vertically for each unit of horizontal change from left to right, as shown in Figure 2.1 and Figure 2.2.



Positive slope, line rises.

FIGURE 2.1



Negative slope, line falls.

FIGURE 2.2

A linear equation that is written in the form $y = mx + b$ is said to be written in **slope-intercept form.**

> **The Slope-Intercept Form of the Equation of a Line**
>
> The graph of the equation
>
> $$y = mx + b$$
>
> is a line whose slope is m and whose y-intercept is $(0, b)$.

Exploration

Use a graphing utility to compare the slopes of the lines $y = mx$ where $m = 0.5$, 1, 2, and 4. Which line rises most quickly? Now, let $m = -0.5$, -1, -2, and -4. Which line falls most quickly? Use a square setting to obtain a true geometric perspective. What can you conclude about the slope and the "rate" at which the line rises or falls?

The icon identifies examples and concepts related to features of the Learning Tools CD-ROM and the *Interactive* and *Internet* versions of this text. For more details see the chart on pages *xix–xxiii*.

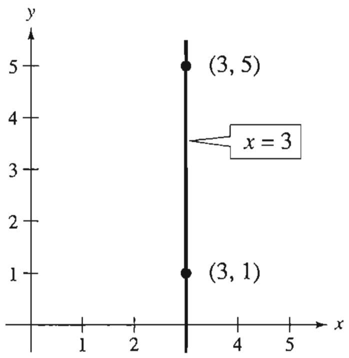


FIGURE 2.3 *Slope is undefined.*

Once you have determined the slope and the y-intercept of a line, it is a relatively simple matter to sketch its graph. In the next example, note that none of the lines is vertical. A vertical line has an equation of the form

$x = a.$ Vertical line

The equation of a vertical line cannot be written in the form $y = mx + b$ because the slope of a vertical line is undefined, as indicated in Figure 2.3.

Example 1 ▶ Graphing a Linear Equation

Sketch the graph of each linear equation.

a. $y = 2x + 1$

b. $y = 2$

c. $x + y = 2$

Solution

a. Because $b = 1$, the y-intercept is $(0, 1)$. Moreover, because the slope is $m = 2$, the line *rises* two units for each unit the line moves to the right, as shown in Figure 2.4.

b. By writing this equation in the form $y = (0)x + 2$, you can see that the y-intercept is $(0, 2)$ and the slope is zero. A zero slope implies that the line is horizontal—that is, it doesn't rise *or* fall, as shown in Figure 2.5.

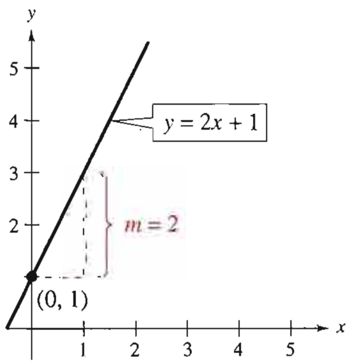
c. By writing this equation in slope-intercept form

$x + y = 2$ Write original equation.

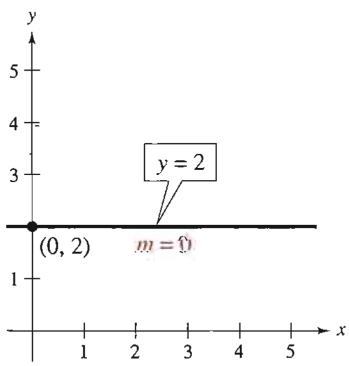
$y = -x + 2$ Subtract x from each side.

$y = (-1)x + 2$ Write in slope-intercept form.

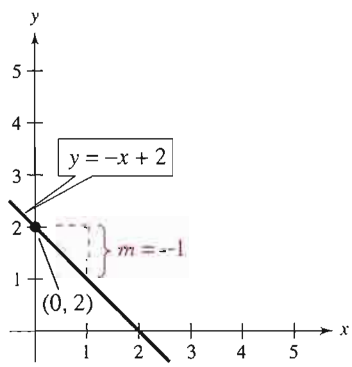
you can see that the y-intercept is $(0, 2)$. Moreover, because the slope is $m = -1$, the line *falls* one unit for each unit the line moves to the right, as shown in Figure 2.6.



When m is positive, the line rises.
FIGURE 2.4



When m is 0, the line is horizontal.
FIGURE 2.5



When m is negative, the line falls.
FIGURE 2.6

In real-life problems, the slope of a line can be interpreted as either a *ratio* or a *rate*. If the x-axis and y-axis have the same unit of measure, then the slope has no units and is a **ratio.** If the x-axis and y-axis have different units of measure, then the slope is a **rate** or **rate of change.**

Example 2 ▶ Using Slope as a Ratio

The *Interactive* CD-ROM and *Internet* versions of this text offer a Try It for each example in the text.

The maximum recommended slope of a wheelchair ramp is $\frac{1}{12}$. A business is installing a wheelchair ramp that rises 22 inches over a horizontal length of 24 feet. Is the ramp steeper than recommended? (Source: Americans with Disabilities Act Handbook)

Solution

The horizontal length of the ramp is 24 feet or $12(24) = 288$ inches, as shown in Figure 2.7. So, the slope of the ramp is

$$\begin{aligned} \text{Slope} &= \frac{\text{vertical change}}{\text{horizontal change}} \\ &= \frac{22 \text{ in.}}{288 \text{ in.}} \\ &\approx 0.076. \end{aligned}$$

Because $\frac{1}{12} \approx 0.083$, the slope of the ramp is not steeper than recommended.

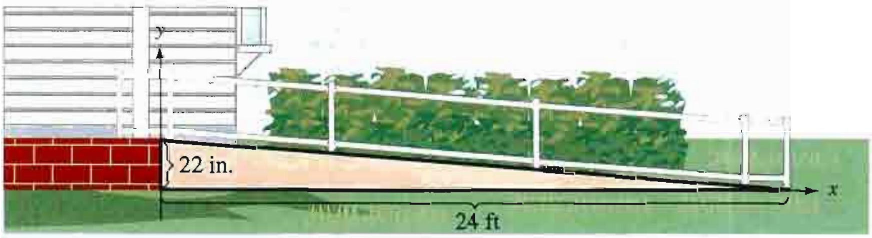


FIGURE 2.7

Example 3 ▶ Using Slope as a Rate of Change

A kitchen appliance manufacturing company determines that the total cost in dollars of producing x units of a blender is

$C = 25x + 3500.$ Cost equation

Describe the practical significance of the y-intercept and slope of this line.

Solution

The y-intercept $(0, 3500)$ tells you that the cost of producing zero units is \$3500. This is the *fixed cost* of production—it includes costs that must be paid regardless of the number of units produced. The slope of $m = 25$ tells you that the cost of producing each unit is \$25, as shown in Figure 2.8. Economists call the cost per unit the *marginal cost*. If the production increases by one unit, then the "margin," or extra amount of cost, is \$25. So, the cost increases at a rate of \$25 per unit.

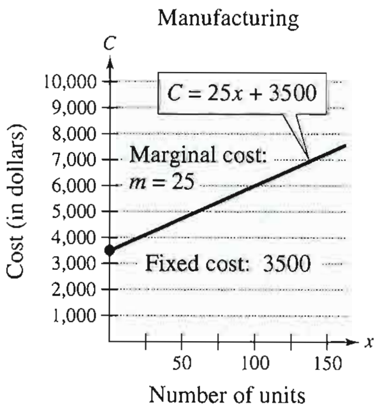


FIGURE 2.8 *Production cost*

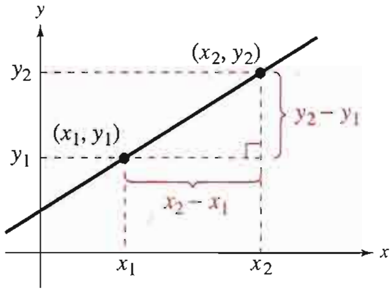


FIGURE 2.9

Finding the Slope of a Line

Given an equation of a line, you can find its slope by writing the equation in slope-intercept form. If you are not given an equation, you can still find the slope of a line. For instance, suppose you want to find the slope of the line passing through the points (x_1, y_1) and (x_2, y_2), as shown in Figure 2.9. As you move from left to right along this line, a change of $(y_2 - y_1)$ units in the vertical direction corresponds to a change of $(x_2 - x_1)$ units in the horizontal direction.

$$y_2 - y_1 = \text{the change in } y = \text{rise}$$

and

$$x_2 - x_1 = \text{the change in } x = \text{run}$$

The ratio of $(y_2 - y_1)$ to $(x_2 - x_1)$ represents the slope of the line that passes through the points (x_1, y_1) and (x_2, y_2).

$$\begin{aligned}\text{Slope} &= \frac{\text{change in } y}{\text{change in } x}\\ &= \frac{\text{rise}}{\text{run}}\\ &= \frac{y_2 - y_1}{x_2 - x_1}\end{aligned}$$

The Slope of a Line Passing Through Two Points

The **slope** m of the nonvertical line through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where $x_1 \neq x_2$.

When this formula is used for slope, the *order of subtraction* is important. Given two points on a line, you are free to label either one of them as (x_1, y_1) and the other as (x_2, y_2). However, once you have done this, you must form the numerator and denominator using the same order of subtraction.

$$\underbrace{m = \frac{y_2 - y_1}{x_2 - x_1}}_{\text{Correct}} \qquad \underbrace{m = \frac{y_1 - y_2}{x_1 - x_2}}_{\text{Correct}} \qquad \underbrace{\cancel{m = \frac{y_2 - y_1}{x_1 - x_2}}}_{\text{Incorrect}}$$

For instance, the slope of the line passing through the points $(3, 4)$ and $(5, 7)$ can be calculated as

$$m = \frac{7 - 4}{5 - 3} = \frac{3}{2}$$

or, reversing the subtraction order in both the numerator and denominator, as

$$m = \frac{4 - 7}{3 - 5} = \frac{-3}{-2} = \frac{3}{2}.$$

The *Interactive* CD-ROM and *Internet* versions of this text offer a Quiz for every section of the text.

Example 4 ▶ Finding the Slope of a Line Through Two Points

Find the slope of the line passing through each pair of points.

a. $(-2,\ 0)$ and $(3, 1)$ **b.** $(-1,\ 2)$ and $(2, 2)$

c. $(0, 4)$ and $(1, -1)$ **d.** $(3, 4)$ and $(3, 1)$

Solution

a. Letting $(x_1,\ y_1) = (-2,\ 0)$ and $(x_2, y_2) = (3, 1)$, you obtain a slope of

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 0}{3 - (-2)} = \frac{1}{5}.$$

See Figure 2.10.

b. The slope of the line passing through $(-1, 2)$ and $(2, 2)$ is

$$m = \frac{2 - 2}{2 - (-1)} = \frac{0}{3} = 0.$$

See Figure 2.11.

c. The slope of the line passing through $(0, 4)$ and $(1, -1)$ is

$$m = \frac{-1 - 4}{1 - 0} = \frac{-5}{1} = -5.$$

See Figure 2.12.

d. The slope of the line passing through $(3, 4)$ and $(3, 1)$ is

$$m = \frac{1 - 4}{3 - 3} = \cancel{\frac{-3}{0}}.$$

See Figure 2.13.

Because division by 0 is undefined, the slope is undefined and the line is vertical.

STUDY TIP

In Figures 2.10 to 2.13, note the relationships between slope and the orientation of the line.

a. Positive slope; line rises from left to right

b. Zero slope; line is horizontal

c. Negative slope; line falls from left to right

d. Undefined slope; line is vertical

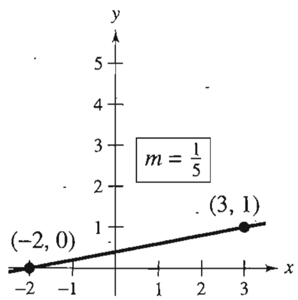


FIGURE 2.10

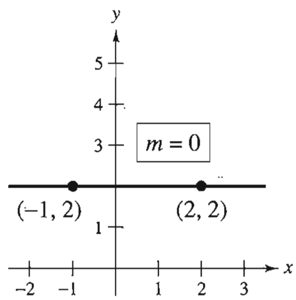


FIGURE 2.11

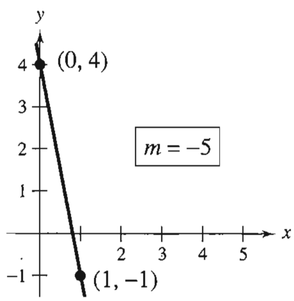


FIGURE 2.12

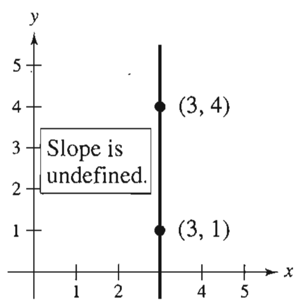


FIGURE 2.13

Writing Linear Equations in Two Variables

If (x_1, y_1) is a point on a line of slope m and (x, y) is *any other* point on the line, then

$$\frac{y - y_1}{x - x_1} = m.$$

This equation, involving the variables x and y, can be rewritten in the form

$$y - y_1 = m(x - x_1)$$

which is the **point-slope form** of the equation of a line.

Point-Slope Form of the Equation of a Line

The equation of the line with slope m passing through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1).$$

The point-slope form is most useful for *finding* the equation of a line. You should remember this form.

Example 5 ▶ Using the Point-Slope Form

Find the slope-intercept form of the equation of the line that has a slope of 3 and passes through the point $(1, -2)$.

Solution

Use the point-slope form with $m = 3$ and $(x_1, y_1) = (1, -2)$.

$$y - y_1 = m(x - x_1) \qquad \text{Point-slope form}$$

$$y - (-2) = 3(x - 1) \qquad \text{Substitute for } m, x_1, \text{ and } y_1.$$

$$y + 2 = 3x - 3 \qquad \text{Simplify.}$$

$$y = 3x - 5 \qquad \text{Write in slope-intercept form.}$$

The slope-intercept form of the equation of the line is $y = 3x - 5$. The graph of this line is shown in Figure 2.14.

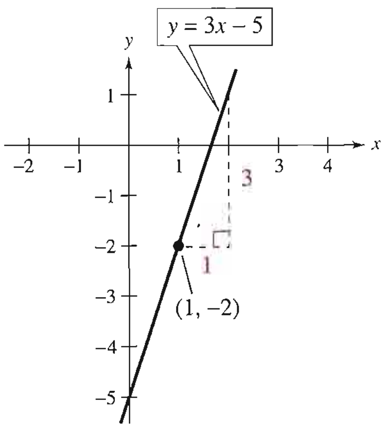


FIGURE 2.14

The point-slope form can be used to find an equation of the line passing through two points (x_1, y_1) and (x_2, y_2). To do this, first find the slope of the line

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \qquad x_1 \neq x_2$$

and then use the point-slope form to obtain the equation

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \qquad \text{Two-point form}$$

This is sometimes called the **two-point form** of the equation of a line.

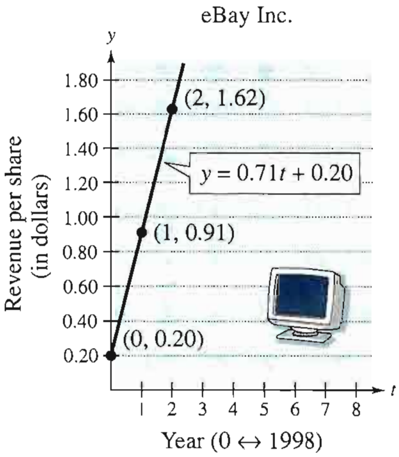


FIGURE 2.15

Example 6 ▶ Predicting Revenue per Share

The revenue per share for eBay Inc. was \$0.20 in 1998 and \$0.91 in 1999. Using only this information, write a linear equation that gives the revenue per share in terms of the year. Then predict the revenue per share for 2000. (Source: eBay Inc.)

Solution

Let $t = 0$ represent 1998. Then the two given values are represented by the data points $(0, 0.20)$ and $(1, 0.91)$. The slope of the line through these points is

$$m = \frac{0.91 - 0.20}{1 - 0}$$

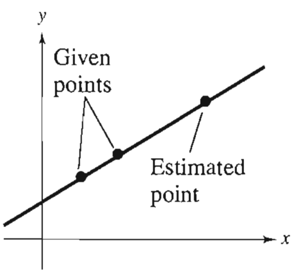
$$= 0.71.$$

Using the point-slope form, you can find the equation that relates the revenue per share y and the year t to be

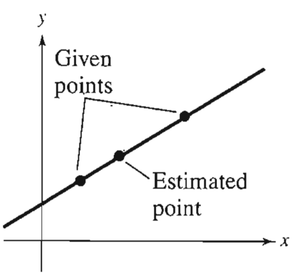
$$y - 0.20 = 0.71(t - 0) \qquad \text{Write in point-slope form.}$$

$$y = 0.71t + 0.20. \qquad \text{Write in slope-intercept form.}$$

According to this equation, the revenue per share in 2000 was \$1.62, as shown in Figure 2.15. (In this case, the prediction is quite good—the actual revenue per share in 2000 was \$1.60.)



Linear extrapolation
FIGURE 2.16



Linear interpolation
FIGURE 2.17

The prediction method illustrated in Example 6 is called **linear extrapolation.** Note in Figure 2.16 that an extrapolated point does not lie between the given points. When the estimated point lies between two given points, as shown in Figure 2.17, the procedure is called **linear interpolation.**

Because the slope of a vertical line is not defined, its equation cannot be written in slope-intercept form. However, every line has an equation that can be written in the **general form**

$$Ax + By + C = 0 \qquad \text{General form}$$

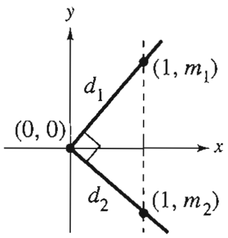
where A and B are not both zero. For instance, the vertical line given by $x = a$ can be represented by the general form $x - a = 0$.

Equations of Lines

1. General form: $Ax + By + C = 0$
2. Vertical line: $x = a$
3. Horizontal line: $y = b$
4. Slope-intercept form: $y = mx + b$
5. Point-slope form: $y - y_1 = m(x - x_1)$
6. Two-point form: $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

Exploration

Find d_1 and d_2 in terms of m_1 and m_2, respectively (see figure). Then use the Pythagorean Theorem to find a relationship between m_1 and m_2.



Parallel and Perpendicular Lines

Slope can be used to decide whether two nonvertical lines in a plane are parallel, perpendicular, or neither.

Parallel and Perpendicular Lines

1. Two distinct nonvertical lines are **parallel** if and only if their slopes are equal. That is, $m_1 = m_2$.
2. Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other. That is, $m_1 = -1/m_2$.

Example 7 ▶ Finding Parallel and Perpendicular Lines

Find the slope-intercept forms of the equations of the lines that pass through the point $(2, -1)$ and are (a) parallel to and (b) perpendicular to the line $2x - 3y = 5$.

Solution

By writing the equation of the given line in slope-intercept form

$2x - 3y = 5$	Write original equation.
$-3y = -2x + 5$	Subtract $2x$ from each side.
$y = \frac{2}{3}x - \frac{5}{3}$	Write in slope-intercept form.

you can see that it has a slope of $m = \frac{2}{3}$, as shown in Figure 2.18.

a. Any line parallel to the given line must also have a slope of $\frac{2}{3}$. So, the line through $(2, -1)$ that is parallel to the given line has the following equation.

$y - (-1) = \frac{2}{3}(x - 2)$	Write in point-slope form.
$3(y + 1) = 2(x - 2)$	Multiply each side by 3.
$3y + 3 = 2x - 4$	Distributive Property
$2x - 3y - 7 = 0$	Write in general form.
$y = \frac{2}{3}x - \frac{7}{3}$	Write in slope-intercept form.

b. Any line perpendicular to the given line must have a slope of $-1/(2/3)$ or $-3/2$. So, the line through $(2, -1)$ that is perpendicular to the given line has the following equation.

$y - (-1) = -\frac{3}{2}(x - 2)$	Write in point-slope form.
$2(y + 1) = -3(x - 2)$	Multiply each side by 2.
$2y + 2 = -3x + 6$	Distributive Property
$3x + 2y - 4 = 0$	Write in general form.
$y = -\frac{3}{2}x + 2$	Write in slope-intercept form.

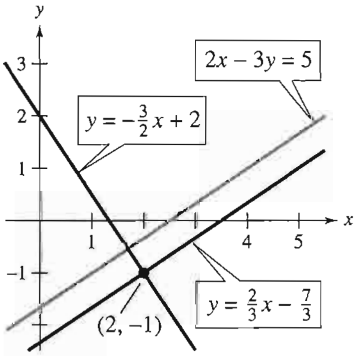


FIGURE 2.18

Technology

On a graphing utility, lines will not appear to have the correct slope unless you use a viewing window that has a square setting. For instance, try graphing the lines in Example 7 using the standard setting $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$. Then reset the viewing window with the square setting $-9 \leq x \leq 9$ and $-6 \leq y \leq 6$. On which setting do the lines $y = \frac{2}{3}x - \frac{5}{3}$ and $y = -\frac{3}{2}x + 2$ appear perpendicular?

Notice in Example 7 how the slope-intercept form is used to obtain information about the graph of a line, whereas the point-slope form is used to write the equation of a line.

Application

Most business expenses can be deducted in the same year they occur. One exception is the cost of property that has a useful life of more than 1 year. Such costs must be *depreciated* over the useful life of the property. If the *same amount* is depreciated each year, the procedure is called *linear* or *straight-line depreciation*. The *book value* is the difference between the original value and the total amount of depreciation accumulated to date.

Example 8 ▶ Straight-Line Depreciation

Your publishing company has purchased a \$12,000 machine that has a useful life of 8 years. The salvage value at the end of 8 years is \$2000. Write a linear equation that describes the book value of the machine each year.

STUDY TIP

In many real-life applications, the two data points that determine the line are often given in a disguised form. Note how the data points are described in Example 8.

Solution

Let V represent the value of the machine at the end of year t. You can represent the initial value of the machine by the data point (0, 12,000) and the salvage value of the machine by the data point (8, 2000). The slope of the line is

$$m = \frac{2000 - 12{,}000}{8 - 0} = -\$1250$$

which represents the annual depreciation in *dollars per year*. Using the point-slope form, you can write the equation of the line as follows.

$$V - 12{,}000 = -1250(t - 0) \qquad \text{Write in point-slope form.}$$

$$V = -1250t + 12{,}000 \qquad \text{Write in slope-intercept form.}$$

The table shows the book value at the end of each year, and the graph of the equation is shown in Figure 2.19.

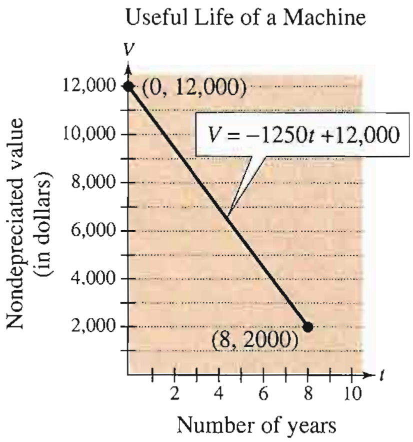


FIGURE 2.19 *Straight-line depreciation*

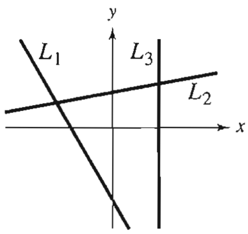
Year, t	Value, V
0	12,000
1	10,750
2	9,500
3	8,250
4	7,000
5	5,750
6	4,500
7	3,250
8	2,000

2.1 Exercises

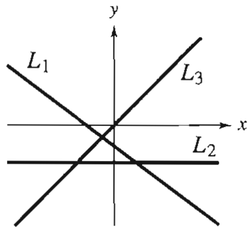
The *Interactive* CD-ROM and *Internet* versions of this text contain step-by-step solutions to all odd-numbered exercises. They also provide Tutorial Exercises for additional help.

In Exercises 1 and 2, identify the line that has each slope.

1. (a) $m = \frac{2}{3}$
(b) m is undefined.
(c) $m = -2$



2. (a) $m = 0$
(b) $m = -\frac{3}{4}$
(c) $m = 1$

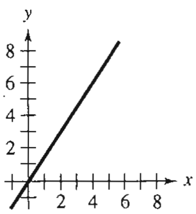


In Exercises 3 and 4, sketch the lines through the point with the indicated slopes on the same set of coordinate axes.

	Point	*Slopes*			
3.	$(2, 3)$	(a) 0	(b) 1	(c) 2	(d) -3
4.	$(-4, 1)$	(a) 3	(b) -3	(c) $\frac{1}{2}$	(d) Undefined

In Exercises 5–8, estimate the slope of the line.

5.



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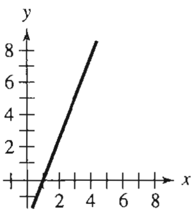

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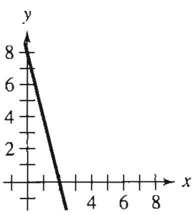

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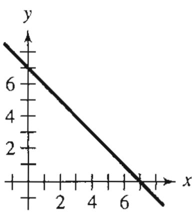


In Exercises 9–20, find the slope and y-intercept (if possible) of the equation of the line. Sketch the line.

9. $y = 5x + 3$
10. $y = x - 10$
11. $y = -\frac{1}{2}x + 4$
12. $y = -\frac{3}{2}x + 6$
13. $5x - 2 = 0$
14. $3y + 5 = 0$
15. $7x + 6y = 30$
16. $2x + 3y = 9$
17. $y - 3 = 0$
18. $y + 4 = 0$
19. $x + 5 = 0$
20. $x - 2 = 0$

In Exercises 21–28, plot the points and find the slope of the line passing through the pair of points.

21. $(-3, -2), (1, 6)$
22. $(2, 4), (4, -4)$
23. $(-6, -1), (-6, 4)$
24. $(0, -10), (-4, 0)$
25. $\left(\frac{11}{2}, -\frac{4}{3}\right), \left(-\frac{3}{2}, -\frac{1}{3}\right)$
26. $\left(\frac{7}{8}, \frac{3}{4}\right), \left(\frac{5}{4}, -\frac{1}{4}\right)$
27. $(4.8, 3.1), (-5.2, 1.6)$
28. $(-1.75, -8.3), (2.25, -2.6)$

In Exercises 29–38, use the point on the line and the slope of the line to find three additional points through which the line passes. (There are many correct answers.)

	Point	*Slope*
29.	$(2, 1)$	$m = 0$
30.	$(-4, 1)$	m is undefined.
31.	$(5, -6)$	$m = 1$
32.	$(10, -6)$	$m = -1$
33.	$(-8, 1)$	m is undefined.
34.	$(-3, -1)$	$m = 0$
35.	$(-5, 4)$	$m = 2$
36.	$(0, -9)$	$m = -2$
37.	$(7, -2)$	$m = \frac{1}{2}$
38.	$(-1, -6)$	$m = -\frac{1}{2}$

In Exercises 39–42, determine whether the lines L_1 and L_2 passing through the pairs of points are parallel, perpendicular, or neither.

39. L_1: $(0, -1), (5, 9)$
L_2: $(0, 3), (4, 1)$

40. L_1: $(-2, -1), (1, 5)$
L_2: $(1, 3), (5, -5)$

41. L_1: $(3, 6), (-6, 0)$
L_2: $(0, -1), \left(5, \frac{7}{3}\right)$

42. L_1: $(4, 8), (-4, 2)$
L_2: $(3, -5), \left(-1, \frac{1}{3}\right)$

43. ***Sales*** The following are the slopes of lines representing annual sales y in terms of time x in years. Use the slopes to interpret any change in annual sales for a one-year increase in time.
(a) The line has a slope of $m = 135$.
(b) The line has a slope of $m = 0$.
(c) The line has a slope of $m = -40$.

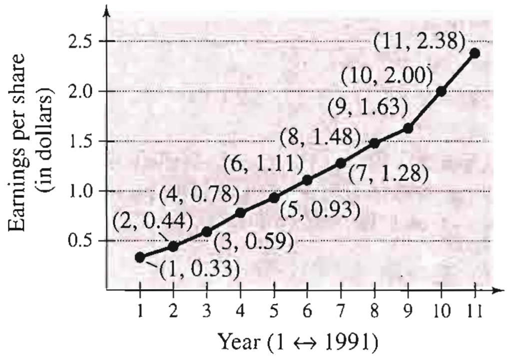
44. ***Revenue*** The following are the slopes of lines representing daily revenues y in terms of time x in days. Use the slopes to interpret any change in daily revenues for a one-day increase in time.

(a) The line has a slope of $m = 400$.

(b) The line has a slope of $m = 100$.

(c) The line has a slope of $m = 0$.

45. ***Earnings per Share*** The graph shows the earnings per share of stock for Auto Zone, Inc. for the years 1991 through 2001. (Source: Auto Zone, Inc.)

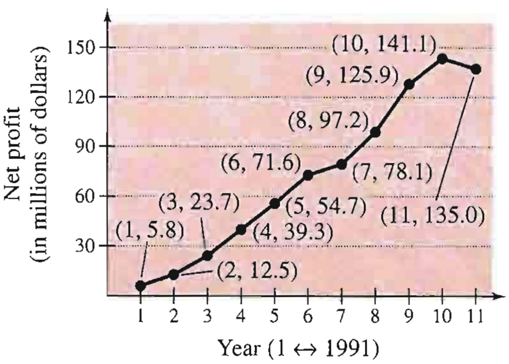


(a) Use the slopes to determine the years in which the earnings per share showed the greatest increase and the smallest increase.

(b) Find the slope of the line segment connecting the years 1991 and 2001.

(c) Interpret the meaning of the slope in part (b) in the context of the problem.

46. ***Net Profit*** The graph shows the net profit (in millions of dollars) for Outback Steakhouse for the years 1991 through 2001. (Source: Outback Steakhouse, Inc.)



(a) Use the slopes to determine the years in which the net profit showed the greatest increase and the smallest increase.

(b) Find the slope of the line segment connecting the years 1991 and 2001.

(c) Interpret the meaning of the slope in part (b) in the context of the problem.

47. ***Road Grade*** From the top of a mountain road, a surveyor takes several horizontal measurements x and several vertical measurements y, as shown in the table (x and y are measured in feet).

x	y
300	−25
600	−50
900	−75
1200	−100
1500	−125
1800	−150
2100	−175

(a) Sketch a scatter plot of the data.

(b) Use a straightedge to sketch the best-fitting line through the points.

(c) Find an equation for the line you sketched in part (b).

(d) Interpret the meaning of the slope of the line in part (c) in the context of the problem.

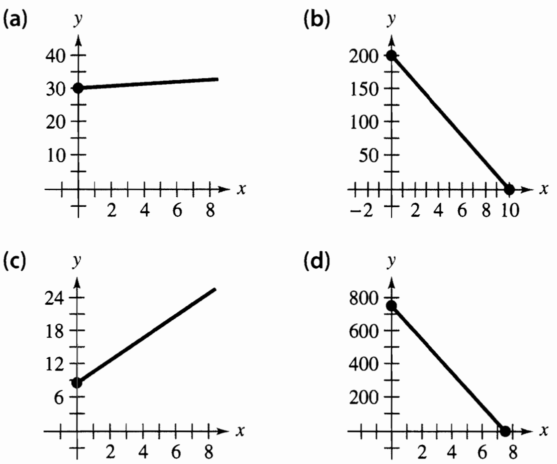
(e) The surveyor needs to put up a road sign that indicates the steepness of the road. For instance, a surveyor would put up a sign that states "8% grade" on a road with a downhill grade that has a slope of $-\frac{8}{100}$. What should the sign state for the road in this problem?

48. ***Road Grade*** You are driving on a road that has a 6% uphill grade. This means that the slope of the road is $\frac{6}{100}$. Approximate the amount of vertical change in your position if you drive 200 feet.

Rate of Change **In Exercises 49 and 50, you are given the dollar value of a product in 2003 and the rate at which the value of the product is expected to change during the next 5 years. Use this information to write a linear equation that gives the dollar value V of the product in terms of the year t. (Let $t = 3$ represent 2003.)**

	2003 Value	*Rate*
49.	\$2540	\$125 increase per year
50.	\$156	\$4.50 increase per year

Graphical Interpretation **In Exercises 51–54, match the description of the situation with its graph. Also determine the slope of each graph and interpret the slope in the context of the situation. [The graphs are labeled (a), (b), (c), and (d).]**



51. A person is paying \$20 per week to a friend to repay a \$200 loan.

52. An employee is paid \$8.50 per hour plus \$2 for each unit produced per hour.

53. A sales representative receives \$30 per day for food plus \$0.32 for each mile traveled.

54. A word processor that was purchased for \$750 depreciates \$100 per year.

In Exercises 55–66, find the slope-intercept form of the equation of the line that passes through the given point and has the indicated slope. Sketch the line.

	Point	*Slope*
55.	$(0, -2)$	$m = 3$
56.	$(0, 10)$	$m = -1$
57.	$(-3, 6)$	$m = -2$
58.	$(0, 0)$	$m = 4$
59.	$(4, 0)$	$m = -\frac{1}{3}$
60.	$(-2, -5)$	$m = \frac{3}{4}$
61.	$(6, -1)$	m is undefined.
62.	$(-10, 4)$	m is undefined.
63.	$\left(4, \frac{5}{2}\right)$	$m = 0$
64.	$\left(-\frac{1}{2}, \frac{3}{2}\right)$	$m = 0$
65.	$(-5.1, 1.8)$	$m = 5$
66.	$(2.3, -8.5)$	$m = -\frac{5}{2}$

In Exercises 67–80, find the slope-intercept form of the equation of the line passing through the points. Sketch the line.

67. $(5, -1), (-5, 5)$

68. $(4, 3), (-4, -4)$

69. $(-8, 1), (-8, 7)$

70. $(-1, 4), (6, 4)$

71. $\left(2, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{5}{4}\right)$

72. $(1, 1), \left(6, -\frac{2}{3}\right)$

73. $\left(-\frac{1}{10}, -\frac{3}{5}\right), \left(\frac{9}{10}, -\frac{9}{5}\right)$

74. $\left(\frac{3}{4}, \frac{3}{2}\right), \left(-\frac{4}{3}, \frac{7}{4}\right)$

75. $(1, 0.6), (-2, -0.6)$

76. $(-8, 0.6), (2, -2.4)$

77. $(2, -1), \left(\frac{1}{3}, -1\right)$

78. $\left(\frac{1}{5}, -2\right), (-6, -2)$

79. $\left(\frac{7}{3}, -8\right), \left(\frac{7}{3}, 1\right)$

80. $(1.5, -2), (1.5, 0.2)$

In Exercises 81–86, use the *intercept form* to find the equation of the line with the given intercepts. The intercept form of the equation of a line with intercepts $(a, 0)$ and $(0, b)$ is

$$\frac{x}{a} + \frac{y}{b} = 1, \quad a \neq 0, \quad b \neq 0.$$

81. x-intercept: $(2, 0)$
y-intercept: $(0, 3)$

82. x-intercept: $(-3, 0)$
y-intercept: $(0, 4)$

83. x-intercept: $\left(-\frac{1}{6}, 0\right)$
y-intercept: $\left(0, -\frac{2}{3}\right)$

84. x-intercept: $\left(\frac{2}{3}, 0\right)$
y-intercept: $(0, -2)$

85. Point on line: $(1, 2)$
x-intercept: $(c, 0)$
y-intercept: $(0, c)$, $\quad c \neq 0$

86. Point on line: $(-3, 4)$
x-intercept: $(d, 0)$
y-intercept: $(0, d)$, $\quad d \neq 0$

In Exercises 87–96, write the slope-intercept forms of the equations of the lines through the given point (a) parallel to the given line and (b) perpendicular to the given line.

	Point	*Line*
87.	$(2, 1)$	$4x - 2y = 3$
88.	$(-3, 2)$	$x + y = 7$
89.	$\left(-\frac{2}{3}, \frac{7}{8}\right)$	$3x + 4y = 7$
90.	$\left(\frac{7}{8}, \frac{3}{4}\right)$	$5x + 3y = 0$
91.	$(-1, 0)$	$y = -3$
92.	$(4, -2)$	$y = 1$
93.	$(2, 5)$	$x = 4$
94.	$(-5, 1)$	$x = -2$
95.	$(2.5, 6.8)$	$x - y = 4$
96.	$(-3.9, -1.4)$	$6x + 2y = 9$

Graphical Interpretation **In Exercises 97–100, identify any relationships that exist among the lines, and then use a graphing utility to graph the three equations in the same viewing window. Adjust the viewing window so that the slope appears visually correct—that is, so that parallel lines appear parallel and perpendicular lines appear to intersect at a right angle.**

97. (a) $y = 2x$ (b) $y = -2x$ (c) $y = \frac{1}{2}x$

98. (a) $y = \frac{2}{3}x$ (b) $y = -\frac{3}{2}x$ (c) $y = \frac{2}{3}x + 2$

99. (a) $y = -\frac{1}{2}x$ (b) $y = -\frac{1}{2}x + 3$ (c) $y = 2x - 4$

100. (a) $y = x - 8$ (b) $y = x + 1$ (c) $y = -x + 3$

In Exercises 101–104, find a relationship between x and y such that (x, y) is equidistant from the two points.

101. $(4, -1), (-2, 3)$

102. $(6, 5), (1, -8)$

103. $\left(3, \frac{5}{2}\right), (-7, 1)$

104. $\left(-\frac{1}{2}, -4\right), \left(\frac{7}{2}, \frac{5}{4}\right)$

105. ***Cash Flow per Share*** The cash flow per share for Timberland Co. was \$0.18 in 1995 and \$3.65 in 2000. Write a linear equation that gives the cash flow per share in terms of the year. Let $t = 0$ represent 1995. Then predict the cash flows for the years 2005 and 2010. (Source: Timberland Co.)

106. ***Number of Stores*** In 1996 there were 3927 J.C. Penney stores and in 2000 there were 3800 stores. Write a linear equation that gives the number of stores in terms of the year. Let $t = 0$ represent 1996. Then predict the numbers of stores for the years 2005 and 2010. (Source: J.C. Penney Co.)

107. ***Annual Salary*** A jeweler's salary was \$28,500 in 1998 and \$32,900 in 2000. The jeweler's salary follows a linear growth pattern. What will the jeweler's salary be in 2005?

108. ***College Enrollment*** Ohio University had 27,913 students in 1999 and 28,197 students in 2001. The enrollment appears to follow a linear growth pattern. How many students will Ohio University have in 2005? (Source: Ohio University)

109. ***Depreciation*** A sub shop purchases a used pizza oven for \$875. After 5 years, the oven will have to be replaced. Write a linear equation giving the value V of the equipment during the 5 years it will be in use.

110. ***Depreciation*** A school district purchases a high-volume printer, copier, and scanner for \$25,000. After 10 years, the equipment will have to be replaced. Its value at that time is expected to be \$2000. Write a linear equation giving the value V of the equipment during the 10 years it will be in use.

111. ***Sales*** A discount outlet is offering a 15% discount on all items. Write a linear equation giving the sale price S for an item with a list price L.

112. ***Hourly Wage*** A microchip manufacturer pays its assembly line workers \$11.50 per hour. In addition, workers receive a piecework rate of \$0.75 per unit produced. Write a linear equation for the hourly wage W in terms of the number of units x produced per hour.

113. ***Cost, Revenue, and Profit*** A roofing contractor purchases a shingle delivery truck with a shingle elevator for \$36,500. The vehicle requires an average expenditure of \$5.25 per hour for fuel and maintenance, and the operator is paid \$11.50 per hour.

(a) Write a linear equation giving the total cost C of operating this equipment for t hours. (Include the purchase cost of the equipment.)

(b) Assuming that customers are charged \$27 per hour of machine use, write an equation for the revenue R derived from t hours of use.

(c) Use the formula for profit $(P = R - C)$ to write an equation for the profit derived from t hours of use.

(d) Use the result of part (c) to find the break-even point—that is, the number of hours this equipment must be used to yield a profit of 0 dollars.

114. ***Rental Demand*** A real estate office handles an apartment complex with 50 units. When the rent per unit is \$580 per month, all 50 units are occupied. However, when the rent is \$625 per month, the average number of occupied units drops to 47. Assume that the relationship between the monthly rent p and the demand x is linear.

(a) Write the equation of the line giving the demand x in terms of the rent p.

(b) Use this equation to predict the number of units occupied when the rent is \$655.

(c) Predict the number of units occupied when the rent is \$595.

115. ***Geometry*** The length and width of a rectangular garden are 15 meters and 10 meters, respectively. A walkway of width x surrounds the garden.

(a) Draw a diagram that gives a visual representation of the problem.

(b) Write the equation for the perimeter y of the walkway in terms of x.

(c) Use a graphing utility to graph the equation for the perimeter.

(d) Determine the slope of the graph in part (c). For each additional one-meter increase in the width of the walkway, determine the increase in its perimeter.

116. ***Monthly Salary*** A pharmaceutical salesperson receives a monthly salary of \$2500 plus a commission of 7% of sales. Write a linear equation for the salesperson's monthly wage W in terms of monthly sales S.

117. ***Business Costs*** A sales representative of a company using a personal car receives \$120 per day for lodging and meals plus \$0.35 per mile driven. Write a linear equation giving the daily cost C to the company in terms of x, the number of miles driven.

118. ***Sports*** The average annual salaries of major league baseball players (in thousands of dollars) from 1995 to 2002 are shown in the scatter plot. Find the equation of the line that you think best fits this data. (Let y represent the average salary and let t represent the year, with $t = 5$ corresponding to 1995.) (Source: The Associated Press)

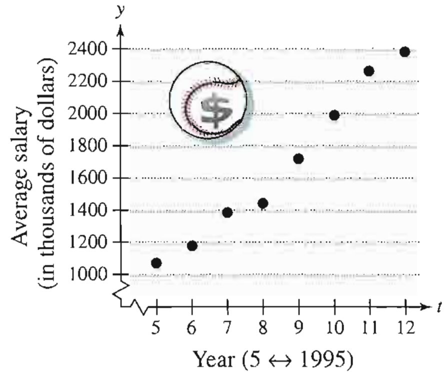


FIGURE FOR 118

▶ Model It

119. ***Data Analysis*** The average monthly cellular phone bills y (in dollars) for subscribers in the United States from 1990 through 1999, where x is the year, are shown as data points (x, y). (Source: Cellular Telecommunications Industry Association)

(1990, 80.90)	(1995, 51.00)
(1991, 72.74)	(1996, 47.70)
(1992, 68.68)	(1997, 42.78)
(1993, 61.48)	(1998, 39.43)
(1994, 56.21)	(1999, 41.24)

(a) Sketch a scatter plot of the data. Let $x = 0$ correspond to 1990.

(b) Sketch the best-fitting line through the points.

(c) Find the equation of the line from part (b). Explain the procedure you used.

(d) Write a short paragraph explaining the meaning of the slope and y-intercept of the line in terms of the data.

(e) Compare the values obtained using your model with the actual values.

(f) Use your model to estimate the average monthly cellular phone bill in 2005.

120. ***Data Analysis*** An instructor gives regular 20-point quizzes and 100-point exams in an algebra course. Average scores for six students, given as data points (x, y) where x is the average quiz score and y is the average test score, are $(18, 87)$, $(10, 55)$, $(19, 96)$, $(16, 79)$, $(13, 76)$, and $(15, 82)$. [*Note:* There are many correct answers for parts (b)–(d).]

(a) Sketch a scatter plot of the data.

(b) Use a straightedge to sketch the best-fitting line through the points.

(c) Find an equation for the line sketched in part (b).

(d) Use the equation in part (c) to estimate the average test score for a person with an average quiz score of 17.

(e) The instructor adds 4 points to the average test score of everyone in the class. Describe the changes in the positions of the plotted points and the change in the equation of the line.

Synthesis

True or False? **In Exercises 121 and 122, determine whether the statement is true or false. Justify your answer.**

121. A line with a slope of $-\frac{5}{7}$ is steeper than a line with a slope of $-\frac{6}{7}$.

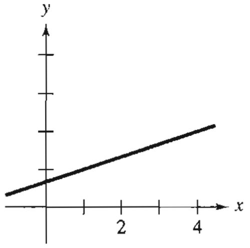
122. The line through $(-8, 2)$ and $(-1, 4)$ and the line through $(0, -4)$ and $(-7, 7)$ are parallel.

123. Explain how you could show that the points $A\,(2, 3)$, $B\,(2, 9)$, and $C\,(4, 3)$ are the vertices of a right triangle.

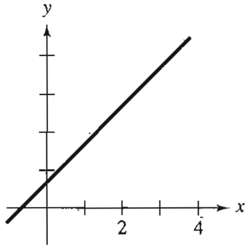
124. Explain why the slope of a vertical line is said to be undefined.

125. With the information given in the graphs, is it possible to determine the slope of each line? Is it possible that the lines could have the same slope? Explain.

(a)



(b)



126. The slopes of two lines are -4 and $\frac{5}{2}$. Which is steeper? Explain.

127. The value V of a molding machine t years after it is purchased is

$$V = -4000t + 58{,}500, \quad 0 \le t \le 5.$$

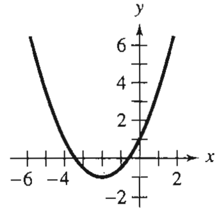
Explain what the V-intercept and slope measure.

128. ***Think About It*** Is it possible for two lines with positive slopes to be perpendicular? Explain.

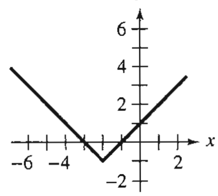
Review

In Exercises 129–132, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]

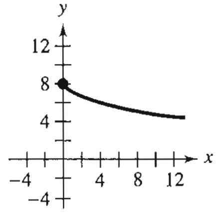
(a)



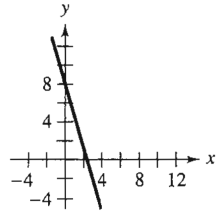
(b)



(c)



(d)



129. $y = 8 - 3x$

130. $y = 8 - \sqrt{x}$

131. $y = \frac{1}{2}x^2 + 2x + 1$

132. $y = |x + 2| - 1$

In Exercises 133–138, find all the solutions of the equation. Check your solution(s) in the original equation.

133. $-7(3 - x) = 14(x - 1)$

134. $\dfrac{8}{2x - 7} = \dfrac{4}{9 - 4x}$

135. $2x^2 - 21x + 49 = 0$

136. $x^2 - 8x + 3 = 0$

137. $\sqrt{x - 9} + 15 = 0$

138. $3x - 16\sqrt{x} + 5 = 0$

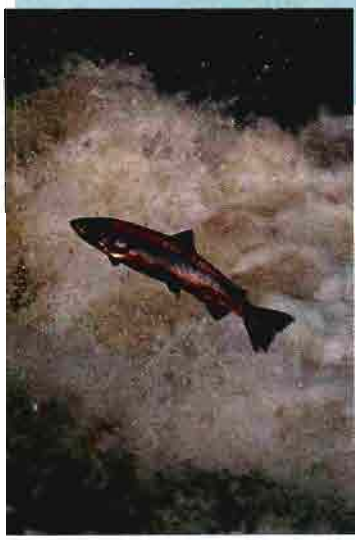
2.2 Functions

▶ What you should learn

- How to determine whether relations between two variables are functions
- How to use function notation and evaluate functions
- How to find the domains of functions
- How to use functions to model and solve real-life problems

▶ Why you should learn it

Functions can be used to model and solve real-life problems. For instance, in Exercise 99 on page 200, you will use a function to find the number of threatened and endangered fish in the world.



B&C Alexander

Introduction to Functions

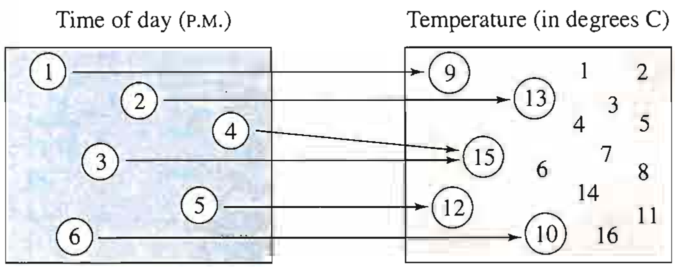
Many everyday phenomena involve two quantities that are related to each other by some rule of correspondence. The mathematical term for such a rule of correspondence is a **relation.** In mathematics, relations are often represented by mathematical equations and formulas. For instance, the simple interest I earned on \$1000 for 1 year is related to the annual interest rate r by the formula $I = 1000r$.

The formula $I = 1000r$ represents a special kind of relation that matches each item from one set with exactly one item from a different set. Such a relation is called a **function.**

Definition of a Function

A **function** f from a set A to a set B is a relation that assigns to each element x in the set A exactly one element y in the set B. The set A is the **domain** (or set of inputs) of the function f, and the set B contains the **range** (or set of outputs).

To help understand this definition, look at the function that relates the time of day to the temperature in Figure 2.20.



Set A is the domain.
Inputs: 1, 2, 3, 4, 5, 6

Set B contains the range.
Outputs: 9, 10, 12, 13, 15

FIGURE 2.20

This function can be represented by the following ordered pairs, in which the first coordinate is the input and the second coordinate is the output.

$$\{(1, 9°), (2, 13°), (3, 15°), (4, 15°), (5, 12°), (6, 10°)\}$$

Characteristics of a Function from Set A to Set B

1. Each element in A must be matched with an element in B.
2. Some elements in B may not be matched with any element in A.
3. Two or more elements in A may be matched with the same element in B.
4. An element in A (the domain) cannot be matched with two different elements of B.

Functions are commonly represented in four ways.

Four Ways to Represent a Function

1. *Verbally* by a sentence that describes how the input variable is related to the output variable
2. *Numerically* by a table or a list of ordered pairs that matches input values with output values
3. *Graphically* by points on a graph in a coordinate plane in which the input values are represented by the horizontal axis and the output values are represented by the vertical axis
4. *Algebraically* by an equation in two variables

To determine whether or not a relation is a function, you must decide whether each input value is matched with exactly one output value. If any input value is matched with two or more output values, the relation is not a function.

Example 1 ▶ Testing for Functions

Determine whether the relation represents y as a function of x.

a. The input value x is the number of representatives from a state, and the output value y is the number of senators.

b.

Input x	Output y
2	11
2	10
3	8
4	5
5	1

c.

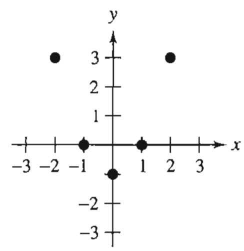


FIGURE 2.21

Solution

a. This verbal description *does* describe y as a function of x. Regardless of the value of x, the value of y is always 2. Such functions are called *constant functions*.

b. This table *does not* describe y as a function of x. The input value 2 is matched with two different y-values.

c. The graph in Figure 2.21 *does* describe y as a function of x. Each input value is matched with exactly one output value.

Representing functions by sets of ordered pairs is common in *discrete mathematics*. In algebra, however, it is more common to represent functions by equations or formulas involving two variables. For instance, the equation

$$y = x^2 \qquad y \text{ is a function of } x.$$

represents the variable y as a function of the variable x. In this equation, x is


The Granger Collection

Historical Note

Leonhard Euler (1707–1783), a Swiss mathematician, is considered to have been the most prolific and productive mathematician in history. One of his greatest influences on mathematics was his use of symbols, or notation. The function notation $y = f(x)$ was introduced by Euler.

the **independent variable** and y is the **dependent variable.** The domain of the function is the set of all values taken on by the independent variable x, and the range of the function is the set of all values taken on by the dependent variable y.

Example 2 ▶ Testing for Functions Represented Algebraically

Which of the equations represent(s) y as a function of x?

a. $x^2 + y = 1$ **b.** $-x + y^2 = 1$

Solution

To determine whether y is a function of x, try to solve for y in terms of x.

a. Solving for y yields

$$x^2 + y = 1 \qquad \text{Write original equation.}$$

$$y = 1 - x^2. \qquad \text{Solve for } y.$$

To each value of x there corresponds exactly one value of y. So, y is a function of x.

b. Solving for y yields

$$-x + y^2 = 1 \qquad \text{Write original equation.}$$

$$y^2 = 1 + x \qquad \text{Add } x \text{ to each side.}$$

$$y = \pm\sqrt{1 + x}. \qquad \text{Solve for } y.$$

The $\pm$ indicates that to a given value of x there correspond two values of y. So, y is not a function of x.

Function Notation

When an equation is used to represent a function, it is convenient to name the function so that it can be referenced easily. For example, you know that the equation $y = 1 - x^2$ describes y as a function of x. Suppose you give this function the name "f." Then you can use the following **function notation.**

Input	*Output*	*Equation*
x	$f(x)$	$f(x) = 1 - x^2$

The symbol $f(x)$ is read as *the value of f at x* or simply *f of x*. The symbol $f(x)$ corresponds to the y-value for a given x. So, you can write $y = f(x)$. Keep in mind that f is the *name* of the function, whereas $f(x)$ is the *value* of the function at x. For instance, the function

$$f(x) = 3 - 2x$$

has *function values* denoted by $f(-1)$, $f(0)$, $f(2)$, and so on. To find these values, substitute the specified input values into the given equation.

For $x = -1$, $\quad f(-1) = 3 - 2(-1) = 3 + 2 = 5.$

For $x = 0$, $\quad f(0) = 3 - 2(0) = 3 - 0 = 3.$

For $x = 2$, $\quad f(2) = 3 - 2(2) = 3 - 4 = -1.$

STUDY TIP

In Example 3, note that $g(x + 2)$ is not equal to $g(x) + g(2)$. In general, $g(u + v) \neq g(u) + g(v)$.

Although f is often used as a convenient function name and x is often used as the independent variable, you can use other letters. For instance,

$$f(x) = x^2 - 4x + 7, \quad f(t) = t^2 - 4t + 7, \quad \text{and} \quad g(s) = s^2 - 4s + 7$$

all define the same function. In fact, the role of the independent variable is that of a "placeholder." Consequently, the function could be described by

$$f(\quad) = (\quad)^2 - 4(\quad) + 7.$$

Example 3 ▶ Evaluating a Function

Let $g(x) = -x^2 + 4x + 1$. Find

a. $g(2)$ **b.** $g(t)$ **c.** $g(x + 2)$.

Solution

a. Replacing x with 2 in $g(x) = -x^2 + 4x + 1$ yields the following.

$$g(2) = -(2)^2 + 4(2) + 1 = -4 + 8 + 1 = 5$$

b. Replacing x with t yields the following.

$$g(t) = -(t)^2 + 4(t) + 1 = -t^2 + 4t + 1$$

c. Replacing x with $x + 2$ yields the following.

$$\begin{aligned} g(x + 2) &= -(x + 2)^2 + 4(x + 2) + 1 \\ &= -(x^2 + 4x + 4) + 4x + 8 + 1 \\ &= -x^2 - 4x - 4 + 4x + 8 + 1 \\ &= -x^2 + 5 \end{aligned}$$

A function defined by two or more equations over a specified domain is called a **piecewise-defined function.**

Example 4 ▶ A Piecewise-Defined Function

Evaluate the function when $x = -1, 0$, and 1.

$$f(x) = \begin{cases} x^2 + 1, & x < 0 \\ x - 1, & x \geq 0 \end{cases}$$

Solution

Because $x = -1$ is less than 0, use $f(x) = x^2 + 1$ to obtain

$$f(-1) = (-1)^2 + 1 = 2.$$

For $x = 0$, use $f(x) = x - 1$ to obtain

$$f(0) = (0) - 1 = -1.$$

For $x = 1$, use $f(x) = x - 1$ to obtain

$$f(1) = (1) - 1 = 0.$$

Technology

Use a graphing utility to graph $y = \sqrt{4 - x^2}$. What is the domain of this function? Then graph $y = \sqrt{x^2 - 4}$. What is the domain of this function? Do the domains of these two functions overlap? If so, for what values?

The Domain of a Function

The domain of a function can be described explicitly or it can be *implied* by the expression used to define the function. The **implied domain** is the set of all real numbers for which the expression is defined. For instance, the function

$$f(x) = \frac{1}{x^2 - 4}$$ Domain excludes x-values that result in division by zero.

has an implied domain that consists of all real x other than $x = \pm 2$. These two values are excluded from the domain because division by zero is undefined. Another common type of implied domain is that used to avoid even roots of negative numbers. For example, the function

$$f(x) = \sqrt{x}$$ Domain excludes x-values that result in even roots of negative numbers.

is defined only for $x \geq 0$. So, its implied domain is the interval $[0, \infty)$. In general, the domain of a function *excludes* values that would cause division by zero *or* that would result in the even root of a negative number.

Example 5 ▶ Finding the Domain of a Function

Find the domain of each function.

a. f: $\{(-3, 0), (-1, 4), (0, 2), (2, 2), (4, -1)\}$ **b.** $g(x) = \dfrac{1}{x + 5}$

c. Volume of a sphere: $V = \frac{4}{3}\pi r^3$ **d.** $h(x) = \sqrt{4 - x^2}$

Solution

a. The domain of f consists of all first coordinates in the set of ordered pairs.

$$\text{Domain} = \{-3, -1, 0, 2, 4\}$$

b. Excluding x-values that yield zero in the denominator, the domain of g is the set of all real numbers $x \neq -5$.

c. Because this function represents the volume of a sphere, the values of the radius r must be positive. So, the domain is the set of all real numbers r such that $r > 0$.

d. This function is defined only for x-values for which

$$4 - x^2 \geq 0.$$

Using the methods described in Section 1.8, you can conclude that $-2 \leq x \leq 2$. So, the domain is the interval $[-2, 2]$.

In Example 5(c), note that the domain of a function may be implied by the physical context. For instance, from the equation

$$V = \tfrac{4}{3}\pi r^3$$

you would have no reason to restrict r to positive values, but the physical context implies that a sphere cannot have a negative radius.

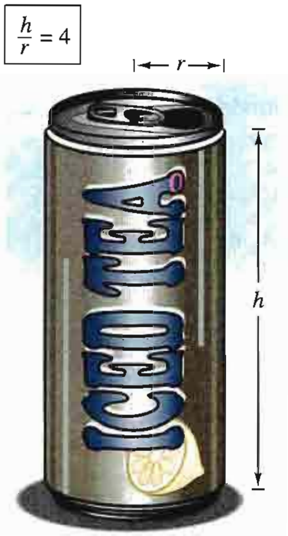


FIGURE 2.22

Applications

Example 6 ▶ The Dimensions of a Container

You work in the marketing department of a soft-drink company and are experimenting with a new can for iced tea that is slightly narrower and taller than a standard can. For your experimental can, the ratio of the height to the radius is 4, as shown in Figure 2.22.

a. Write the volume of the can as a function of the radius r.

b. Write the volume of the can as a function of the height h.

Solution

a. $V(r) = \pi r^2 h = \pi r^2(4r) = 4\pi r^3$ Write V as a function of r.

b. $V(h) = \pi\left(\frac{h}{4}\right)^2 h = \frac{\pi h^3}{16}$ Write V as a function of h.

Example 7 ▶ The Path of a Baseball

A baseball is hit at a point 3 feet above ground at a velocity of 100 feet per second and an angle of 45°. The path of the baseball is given by the function

$$f(x) = -0.0032x^2 + x + 3$$

where y and x are measured in feet, as shown in Figure 2.23. Will the baseball clear a 10-foot fence located 300 feet from home plate?

Solution

When $x = 300$, the height of the baseball is

$$f(300) = -0.0032(300)^2 + 300 + 3$$
$$= 15 \text{ feet.}$$

So, the ball will clear the fence.

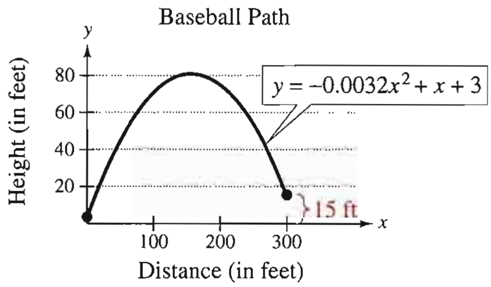


FIGURE 2.23

In the equation in Example 7, the height of the baseball is a function of the distance from home plate.

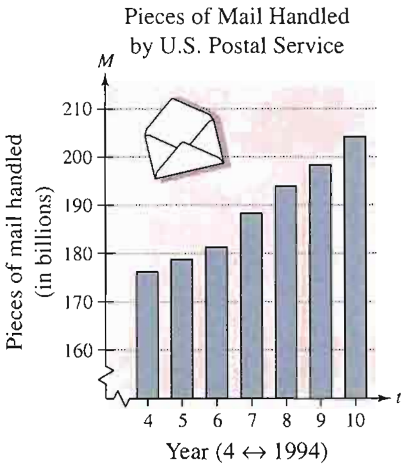


FIGURE 2.24

Example 8 ▶ Pieces of Mail Handled

The number M (in billions) of pieces of mail handled by the U.S. Postal Service increased in a linear pattern from 1994 to 1996, as shown in Figure 2.24. Then, in 1997, the number handled took a jump and, until 2000, increased in a *different* linear pattern. These two patterns can be approximated by the function

$$M(t) = \begin{cases} 167.2 + 2.70t, & 4 \le t \le 6 \\ 152.0 + 5.57t, & 7 \le t \le 10 \end{cases}$$

where $t = 4$ represents 1994. Use this function to approximate the total number of pieces of mail handled from 1994 to 2000. (Source: U.S. Postal Service)

Solution

From 1994 to 1996, use $M(t) = 167.2 + 2.70t$.

$$\underbrace{178.0,}_{1994} \quad \underbrace{180.7,}_{1995} \quad \underbrace{183.4}_{1996}$$

From 1997 to 2000, use $M(t) = 152.0 + 5.57t$.

$$\underbrace{191.0,}_{1997} \quad \underbrace{196.6,}_{1998} \quad \underbrace{202.1,}_{1999} \quad \underbrace{207.7}_{2000}$$

The total of these seven amounts is 1339.5, which implies that the total number of pieces of mail handled was approximately 1.3 trillion.

One of the basic definitions in calculus employs the ratio

$$\frac{f(x+h) - f(x)}{h}, \qquad h \neq 0.$$

This ratio is called a **difference quotient,** as illustrated in Example 9.

Example 9 ▶ Evaluating a Difference Quotient

For $f(x) = x^2 - 4x + 7$, find $\dfrac{f(x+h) - f(x)}{h}$.

Solution

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{[(x+h)^2 - 4(x+h) + 7] - (x^2 - 4x + 7)}{h} \\ &= \frac{x^2 + 2xh + h^2 - 4x - 4h + 7 - x^2 + 4x - 7}{h} \\ &= \frac{2xh + h^2 - 4h}{h} = \frac{h(2x + h - 4)}{h} = 2x + h - 4, \ h \neq 0 \end{aligned}$$

The symbol ∫ indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

Summary of Function Terminology

Function: A **function** is a relationship between two variables such that to each value of the independent variable there corresponds exactly one value of the dependent variable.

Function Notation: $y = f(x)$

f is the *name* of the function.

y is the **dependent variable.**

x is the **independent variable.**

$f(x)$ is the *value of the function at x.*

Domain: The **domain** of a function is the set of all values (inputs) of the independent variable for which the function is defined. If x is in the domain of f, f is said to be *defined* at x. If x is not in the domain of f, f is said to be *undefined* at x.

Range: The range of a function is the set of all values (outputs) assumed by the dependent variable (that is, the set of all function values).

Implied Domain: If f is defined by an algebraic expression and the domain is not specified, the **implied domain** consists of all real numbers for which the expression is defined.

Writing ABOUT MATHEMATICS

Everyday Functions In groups of two or three, identify common real-life functions. Consider everyday activities, events, and expenses, such as long distance telephone calls and car insurance. Here are two examples.

a. The statement, "Your happiness is a function of the grade you receive in this course" *is not* a correct mathematical use of the word "function." The word "happiness" is ambiguous.

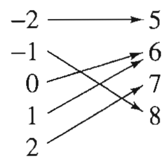
b. The statement, "Your federal income tax is a function of your adjusted gross income" *is* a correct mathematical use of the word "function." Once you have determined your adjusted gross income, your income tax can be determined.

Describe your functions in words. Avoid using ambiguous words. Can you find an example of a piecewise-defined function?

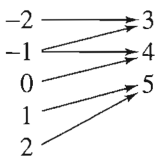
2.2 Exercises

In Exercises 1–4, is the relationship a function?

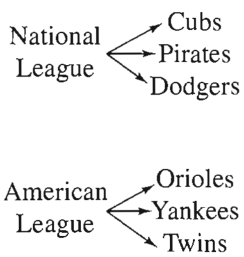
1. *Domain* *Range*



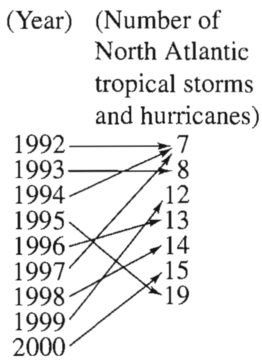
2. *Domain* *Range*



3. *Domain* *Range*



4. *Domain* *Range*



In Exercises 5–8, does the table describe a function? Explain your reasoning.

5.

Input value	-2	-1	0	1	2
Output value	-8	-1	0	1	8

6.

Input value	0	1	2	1	0
Output value	-4	-2	0	2	4

7.

Input value	10	7	4	7	10
Output value	3	6	9	12	15

8.

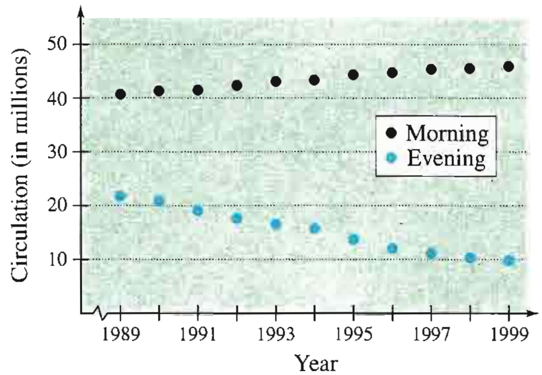
Input value	0	3	9	12	15
Output value	3	3	3	3	3

In Exercises 9 and 10, which sets of ordered pairs represent functions from A to B? Explain.

9. $A = \{0, 1, 2, 3\}$ and $B = \{-2, -1, 0, 1, 2\}$
   (a) $\{(0, 1), (1, -2), (2, 0), (3, 2)\}$
   (b) $\{(0, -1), (2, 2), (1, -2), (3, 0), (1, 1)\}$
   (c) $\{(0, 0), (1, 0), (2, 0), (3, 0)\}$
   (d) $\{(0, 2), (3, 0), (1, 1)\}$

10. $A = \{a, b, c\}$ and $B = \{0, 1, 2, 3\}$
   (a) $\{(a, 1), (c, 2), (c, 3), (b, 3)\}$
   (b) $\{(a, 1), (b, 2), (c, 3)\}$
   (c) $\{(1, a), (0, a), (2, c), (3, b)\}$
   (d) $\{(c, 0), (b, 0), (a, 3)\}$

Circulation of Newspapers **In Exercises 11 and 12, use the graph, which shows the circulation (in millions) of daily newspapers in the United States.** (Source: Editor & Publisher Company)



11. Is the circulation of morning newspapers a function of the year? Is the circulation of evening newspapers a function of the year? Explain.

12. Let $f(x)$ represent the circulation of evening newspapers in year x. Find $f(1998)$.

In Exercises 13–22, determine whether the equation represents y as a function of x.

13. $x^2 + y^2 = 4$
14. $x = y^2$
15. $x^2 + y = 4$
16. $x + y^2 = 4$
17. $2x + 3y = 4$
18. $(x - 2)^2 + y^2 = 4$
19. $y^2 = x^2 - 1$
20. $y = \sqrt{x + 5}$
21. $y = |4 - x|$
22. $|y| = 4 - x$

In Exercises 23–36, evaluate the function at each specified value of the independent variable and simplify.

23. $f(x) = 2x - 3$
(a) $f(1)$ (b) $f(-3)$ (c) $f(x - 1)$

24. $g(y) = 7 - 3y$
(a) $g(0)$ (b) $g\left(\frac{7}{3}\right)$ (c) $g(s + 2)$

25. $V(r) = \frac{4}{3}\pi r^3$
(a) $V(3)$ (b) $V\left(\frac{3}{2}\right)$ (c) $V(2r)$

26. $h(t) = t^2 - 2t$
(a) $h(2)$ (b) $h(1.5)$ (c) $h(x + 2)$

27. $f(y) = 3 - \sqrt{y}$
(a) $f(4)$ (b) $f(0.25)$ (c) $f(4x^2)$

28. $f(x) = \sqrt{x + 8} + 2$
(a) $f(-8)$ (b) $f(1)$ (c) $f(x - 8)$

29. $q(x) = \dfrac{1}{x^2 - 9}$
(a) $q(0)$ (b) $q(3)$ (c) $q(y + 3)$

30. $q(t) = \dfrac{2t^2 + 3}{t^2}$
(a) $q(2)$ (b) $q(0)$ (c) $q(-x)$

31. $f(x) = \dfrac{|x|}{x}$
(a) $f(2)$ (b) $f(-2)$ (c) $f(x - 1)$

32. $f(x) = |x| + 4$
(a) $f(2)$ (b) $f(-2)$ (c) $f(x^2)$

33. $f(x) = \begin{cases} 2x + 1, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$
(a) $f(-1)$ (b) $f(0)$ (c) $f(2)$

34. $f(x) = \begin{cases} x^2 + 2, & x \leq 1 \\ 2x^2 + 2, & x > 1 \end{cases}$
(a) $f(-2)$ (b) $f(1)$ (c) $f(2)$

35. $f(x) = \begin{cases} 3x - 1, & x < -1 \\ 4, & -1 \leq x \leq 1 \\ x^2, & x > 1 \end{cases}$
(a) $f(-2)$ (b) $f\left(-\frac{1}{2}\right)$ (c) $f(3)$

36. $f(x) = \begin{cases} 4 - 5x, & x \leq -2 \\ 0, & -2 < x \leq 2 \\ x^2 + 1, & x > 2 \end{cases}$
(a) $f(-3)$ (b) $f(4)$ (c) $f(-1)$

In Exercises 37–42, complete the table.

37. $f(x) = x^2 - 3$

x	$f(x)$
-2	
-1	
0	
1	
2	

38. $g(x) = \sqrt{x - 3}$

x	$g(x)$
3	
4	
5	
6	
7	

39. $h(t) = \frac{1}{2}|t + 3|$

t	$h(t)$
-5	
-4	
-3	
-2	
-1	

40. $f(s) = \dfrac{|s - 2|}{s - 2}$

s	$f(s)$
0	
1	
$\frac{3}{2}$	
$\frac{5}{2}$	
4	

41. $f(x) = \begin{cases} -\frac{1}{2}x + 4, & x \leq 0 \\ (x - 2)^2, & x > 0 \end{cases}$

x	$f(x)$
-2	
-1	
0	
1	
2	

42. $h(x) = \begin{cases} 9 - x^2, & x < 3 \\ x - 3, & x \geq 3 \end{cases}$

x	$h(x)$
1	
2	
3	
4	
5	

In Exercises 43–50, find all real values of x such that $f(x) = 0$.

43. $f(x) = 15 - 3x$
44. $f(x) = 5x + 1$
45. $f(x) = \dfrac{3x - 4}{5}$
46. $f(x) = \dfrac{12 - x^2}{5}$
47. $f(x) = x^2 - 9$
48. $f(x) = x^2 - 8x + 15$
49. $f(x) = x^3 - x$
50. $f(x) = x^3 - x^2 - 4x + 4$

In Exercises 51–54, find the value(s) of x for which $f(x) = g(x)$.

51. $f(x) = x^2 + 2x + 1, \quad g(x) = 3x + 3$
52. $f(x) = x^4 - 2x^2, \quad g(x) = 2x^2$
53. $f(x) = \sqrt{3x} + 1, \quad g(x) = x + 1$
54. $f(x) = \sqrt{x} - 4, \quad g(x) = 2 - x$

In Exercises 55–68, find the domain of the function.

55. $f(x) = 5x^2 + 2x - 1$
56. $g(x) = 1 - 2x^2$
57. $h(t) = \dfrac{4}{t}$
58. $s(y) = \dfrac{3y}{y + 5}$
59. $g(y) = \sqrt{y - 10}$
60. $f(t) = \sqrt[3]{t + 4}$
61. $f(x) = \sqrt[4]{1 - x^2}$
62. $f(x) = \sqrt[4]{x^2 + 3x}$
63. $g(x) = \dfrac{1}{x} - \dfrac{3}{x + 2}$
64. $h(x) = \dfrac{10}{x^2 - 2x}$
65. $f(s) = \dfrac{\sqrt{s - 1}}{s - 4}$
66. $f(x) = \dfrac{\sqrt{x + 6}}{6 + x}$
67. $f(x) = \dfrac{x - 4}{\sqrt{x}}$
68. $f(x) = \dfrac{x - 5}{\sqrt{x^2 - 9}}$

In Exercises 69–72, assume that the domain of f is the set $A = \{-2, -1, 0, 1, 2\}$. Determine the set of ordered pairs that represents the function f.

69. $f(x) = x^2$
70. $f(x) = x^2 - 3$
71. $f(x) = |x| + 2$
72. $f(x) = |x + 1|$

Exploration In Exercises 73–76, match the data with one of the following functions

$f(x) = cx$, $g(x) = cx^2$, $h(x) = c\sqrt{|x|}$, and $r(x) = \dfrac{c}{x}$

and determine the value of the constant c that will make the function fit the data in the table.

73.

x	y
-4	-32
-1	-2
0	0
1	-2
4	-32

74.

x	y
-4	-1
-1	$-\frac{1}{4}$
0	0
1	$\frac{1}{4}$
4	1

75.

x	y
-4	-8
-1	-32
0	Undef.
1	32
4	8

76.

x	y
-4	6
-1	3
0	0
1	3
4	6

In Exercises 77–84, find the difference quotient and simplify your answer.

77. $f(x) = x^2 - x + 1, \quad \dfrac{f(2 + h) - f(2)}{h}, h \neq 0$
78. $f(x) = 5x - x^2, \quad \dfrac{f(5 + h) - f(5)}{h}, h \neq 0$
79. $f(x) = x^3 + 3x, \quad \dfrac{f(x + h) - f(x)}{h}, h \neq 0$
80. $f(x) = 4x^2 - 2x, \quad \dfrac{f(x + h) - f(x)}{h}, h \neq 0$
81. $g(x) = \dfrac{1}{x^2}, \quad \dfrac{g(x) - g(3)}{x - 3}, x \neq 3$
82. $f(t) = \dfrac{1}{t - 2}, \quad \dfrac{f(t) - f(1)}{t - 1}, t \neq 1$
83. $f(x) = \sqrt{5x}, \quad \dfrac{f(x) - f(5)}{x - 5}, x \neq 5$
84. $f(x) = x^{2/3} + 1, \quad \dfrac{f(x) - f(8)}{x - 8}, x \neq 8$

The symbol $\int$ indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

85. ***Geometry*** Write the area A of a square as a function of its perimeter P.

86. ***Geometry*** Write the area A of a circle as a function of its circumference C.

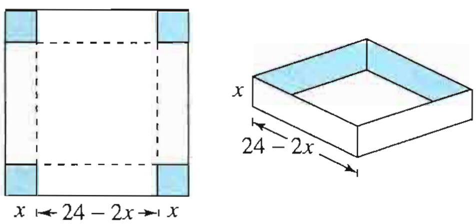
87. ***Maximum Volume*** An open box of maximum volume is to be made from a square piece of material 24 centimeters on a side by cutting equal squares from the corners and turning up the sides (see figure).

(a) The table shows the volume V (in cubic centimeters) of the box for various heights x (in centimeters). Use the table to estimate the maximum volume.

Height, x	Volume, V
1	484
2	800
3	972
4	1024
5	980
6	864

(b) Plot the points (x, V). Does the relation defined by the ordered pairs represent V as a function of x?

(c) If V is a function of x, write the function and determine its domain.



88. ***Maximum Profit*** The cost per unit in the production of a radio model is \$60. The manufacturer charges \$90 per unit for orders of 100 or less. To encourage large orders, the manufacturer reduces the charge by \$0.15 per radio for each unit ordered in excess of 100 (for example, there would be a charge of \$87 per radio for an order size of 120).

(a) The table shows the profit P (in dollars) for various numbers of units ordered, x. Use the table to estimate the maximum profit.

Units, x	Profit, P
110	3135
120	3240
130	3315
140	3360
150	3375
160	3360
170	3315

(b) Plot the points (x, P). Does the relation defined by the ordered pairs represent P as a function of x?

(c) If P is a function of x, write the function and determine its domain.

89. ***Geometry*** A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point $(2, 1)$ (see figure). Write the area A of the triangle as a function of x, and determine the domain of the function.

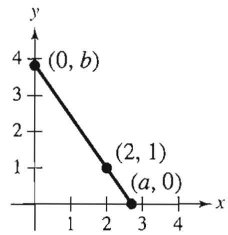


FIGURE FOR 89

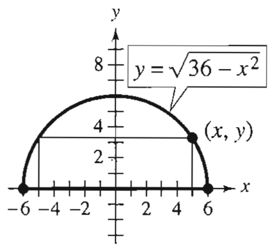


FIGURE FOR 90

90. ***Geometry*** A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{36 - x^2}$ (see figure). Write the area A of the rectangle as a function of x, and determine the domain of the function.

91. ***Average Price*** The average price p (in thousands of dollars) of a new mobile home in the United States from 1990 to 1999 (see figure) can be approximated by the model

$$p(t) = \begin{cases} 0.543t^2 - 0.75t + 27.8, & 0 \le t \le 4 \\ 1.89t + 27.1, & 5 \le t \le 9 \end{cases}$$

where $t = 0$ represents 1990. Use this model to find the average prices of a mobile home in 1990, 1994, 1996, and 1999. (Source: U.S. Census Bureau)

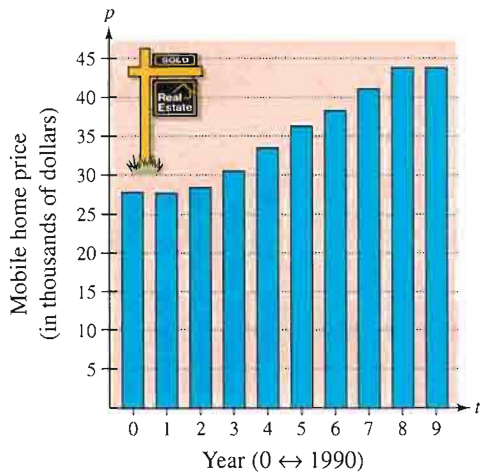


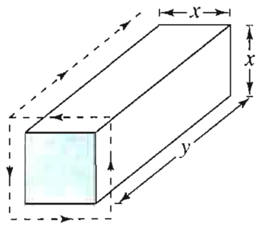
FIGURE FOR 91

92. ***Postal Regulations*** A rectangular package to be sent by the U.S. Postal Service can have a maximum combined length and girth (perimeter of a cross section) of 108 inches (see figure).

(a) Write the volume V of the package as a function of x. What is the domain of the function?

(b) Use a graphing utility to graph your function. Be sure to use the appropriate window setting.

(c) What dimensions will maximize the volume of the package? Explain your answer.



93. ***Cost, Revenue, and Profit*** A company produces a product for which the variable cost is \$12.30 per unit and the fixed costs are \$98,000. The product sells for \$17.98. Let x be the number of units produced and sold.

(a) The total cost for a business is the sum of the variable cost and the fixed costs. Write the total cost C as a function of the number of units produced.

(b) Write the revenue R as a function of the number of units sold.

(c) Write the profit P as a function of the number of units sold. (*Note:* $P = R - C$.)

94. ***Average Cost*** The inventor of a new game believes that the variable cost for producing the game is \$0.95 per unit and the fixed costs are \$6000. The inventor sells each game for \$1.69. Let x be the number of games sold.

(a) The total cost for a business is the sum of the variable cost and the fixed costs. Write the total cost C as a function of the number of games sold.

(b) Write the average cost per unit $\overline{C} = C/x$ as a function of x.

95. ***Transportation*** For groups of 80 or more people, a charter bus company determines the rate per person according to the formula

$$\text{Rate} = 8 - 0.05(n - 80), \qquad n \geq 80$$

where the rate is given in dollars and n is the number of people.

(a) Write the revenue R for the bus company as a function of n.

(b) Use the function in part (a) to complete the table. What can you conclude?

n	90	100	110	120	130	140	150
$R(n)$							

96. ***Physics*** The force F (in tons) of water against the face of a dam is estimated by the function $F(y) = 149.76\sqrt{10}y^{5/2}$, where y is the depth of the water in feet.

(a) Complete the table. What can you conclude from the table?

y	5	10	20	30	40
$F(y)$					

(b) Use the table to approximate the depth at which the force against the dam is 1,000,000 tons.

(c) Find the depth at which the force against the dam is 1,000,000 tons algebraically.

97. ***Height of a Balloon*** A balloon carrying a transmitter ascends vertically from a point 3000 feet from the receiving station.

(a) Draw a diagram that gives a visual representation of the problem. Let h represent the height of the balloon and let d represent the distance between the balloon and the receiving station.

(b) Write the height of the balloon as a function of d. What is the domain of the function?

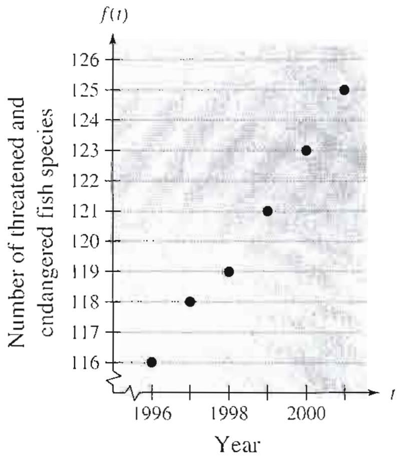
98. ***Path of a Ball*** The height y (in feet) of a baseball thrown by a child is

$$y = -\frac{1}{10}x^2 + 3x + 6$$

where x is the horizontal distance (in feet) from where the ball was thrown. Will the ball fly over the head of another child 30 feet away trying to catch the ball? (Assume that the child who is trying to catch the ball holds a baseball glove at a height of 5 feet.)

▶ Model It

99. ***Wildlife*** The graph shows the number of threatened and endangered fish species in the world from 1996 through 2001. Let $f(t)$ represent the number of threatened and endangered fish species in the year t. (Source: U.S. Fish and Wildlife Service)



(a) Find

$$\frac{f(2001) - f(1996)}{2001 - 1996}$$

and interpret the result in the context of the problem.

(b) Find a linear model for the data algebraically. Let N represent the number of threatened and endangered fish species and let $x = 6$ correspond to 1996.

(c) Use the model found in part (b) to complete the table.

▶ Model It *(continued)*

x	N
6	
7	
8	
9	
10	
11	

(d) Compare your results from part (c) with the actual data.

(e) Use a graphing utility to find a linear model for the data. Let $x = 6$ correspond to 1996. How does the model you found in part (b) compare with the model given by the graphing utility?

Synthesis

True or False? **In Exercises 100 and 101, determine whether the statement is true or false. Justify your answer.**

100. The domain of the function $f(x) = x^4 - 1$ is $(-\infty, \infty)$, and the range of $f(x)$ is $(0, \infty)$.

101. The set of ordered pairs $\{(-8, -2), (-6, 0), (-4, 0), (-2, 2), (0, 4), (2, -2)\}$ represents a function.

102. ***Writing*** In your own words, explain the meanings of *domain* and *range*.

Review

In Exercises 103–106, solve the equation.

103. $\dfrac{t}{3} + \dfrac{t}{5} = 1$

104. $\dfrac{3}{t} + \dfrac{5}{t} = 1$

105. $\dfrac{3}{x(x+1)} - \dfrac{4}{x} = \dfrac{1}{x+1}$

106. $\dfrac{12}{x} - 3 = \dfrac{4}{x} + 9$

In Exercises 107–110, find the equation of the line passing through the pair of points.

107. $(-2, -5), (4, -1)$

108. $(10, 0), (1, 9)$

109. $(-6, 5), (3, -5)$

110. $\left(-\frac{1}{2}, 3\right), \left(\frac{11}{2}, -\frac{1}{3}\right)$

2.3 Analyzing Graphs of Functions

▶ **What you should learn**

- How to use the Vertical Line Test for functions
- How to find the zeros of functions
- How to determine intervals on which functions are increasing or decreasing
- How to identify even and odd functions

▶ **Why you should learn it**

Graphs of functions can help you visualize relationships between variables in real life. For instance, in Exercise 76 on page 209, you will use the graph of a function to represent visually the merchandise trade balance for the United States.

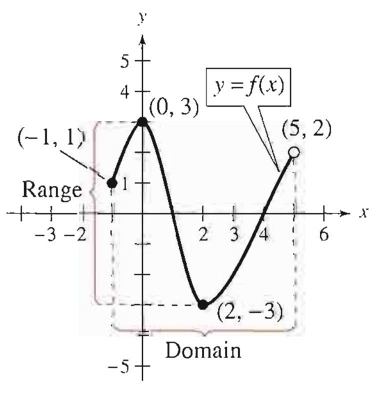


FIGURE 2.26

The Graph of a Function

In Section 2.2, you studied functions from an algebraic point of view. In this section, you will study functions from a graphical perspective.

The **graph of a function** f is the collection of ordered pairs $(x, f(x))$ such that x is in the domain of f. As you study this section, remember that

$$x = \text{the directed distance from the } y\text{-axis}$$

$$f(x) = \text{the directed distance from the } x\text{-axis}$$

as shown in Figure 2.25.

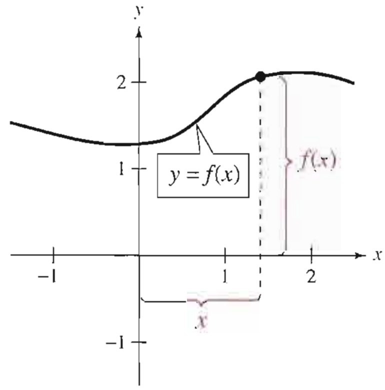


FIGURE 2.25

Example 1 ▶ **Finding the Domain and Range of a Function**

Use the graph of the function f, shown in Figure 2.26, to find (a) the domain of f, (b) the function values $f(-1)$ and $f(2)$, and (c) the range of f.

Solution

a. The closed dot at $(-1, 1)$ indicates that $x = -1$ is in the domain of f, whereas the open dot at $(5, 2)$ indicates $x = 5$ is not in the domain. So, the domain of f is all x in the interval $[-1, 5)$.

b. Because $(-1, 1)$ is a point on the graph of f, it follows that $f(-1) = 1$. Similarly, because $(2, -3)$ is a point on the graph of f, it follows that $f(2) = -3$.

c. Because the graph does not extend below $f(2) = -3$ or above $f(0) = 3$, the range of f is the interval $[-3, 3]$.

The use of dots (open or closed) at the extreme left and right points of a graph indicates that the graph does not extend beyond these points. If no such dots are shown, assume that the graph extends beyond these points.

By the definition of a function, at most one y-value corresponds to a given x-value. This means that the graph of a function cannot have two or more different points with the same x-coordinate, and no two points on the graph of a function can be vertically above and below each other. It follows, then, that a vertical line can intersect the graph of a function at most once. This observation provides a convenient visual test called the **Vertical Line Test** for functions.

Vertical Line Test for Functions

A set of points in a coordinate plane is the graph of y as a function of x if and only if no *vertical* line intersects the graph at more than one point.

Example 2 ▶ Vertical Line Test for Functions

Use the Vertical Line Test to decide whether the graphs in Figure 2.27 represent y as a function of x.

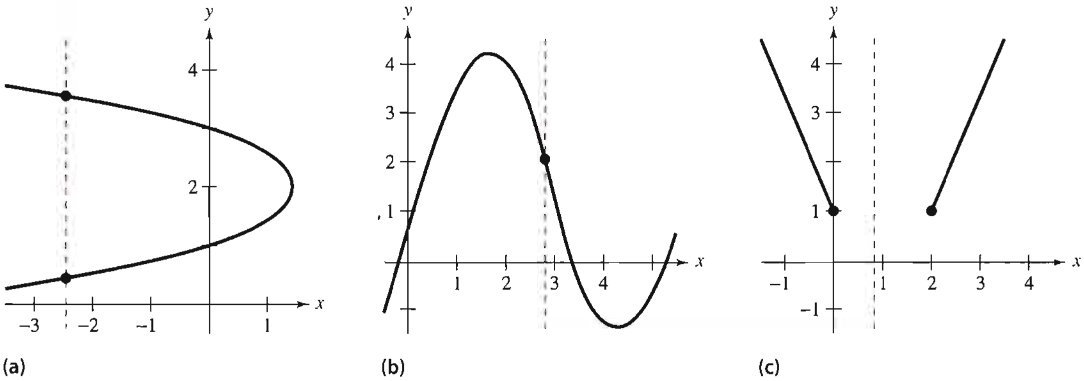


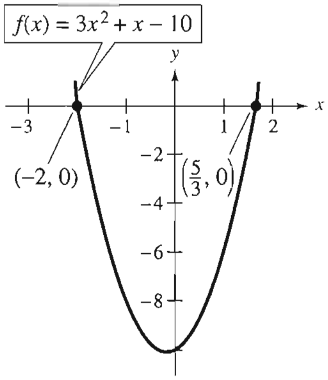
FIGURE 2.27

Solution

a. This *is not* a graph of y as a function of x, because you can find a vertical line that intersects the graph twice. That is, for a particular input x, there is more than one output y.

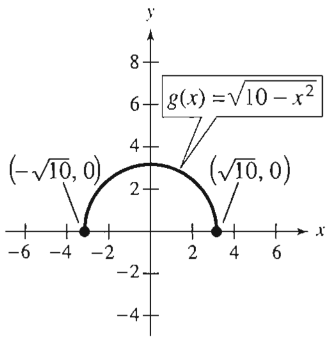
b. This *is* a graph of y as a function of x, because every vertical line intersects the graph at most once. That is, for a particular input x, there is at most one output y.

c. This *is* a graph of y as a function of x. (Note that if a vertical line does not intersect the graph, it simply means that the function is undefined for that particular value of x.) That is, for a particular input x, there is at most one output y.



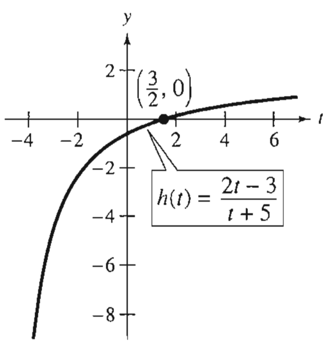
Zeros of f: $x = -2, x = \frac{5}{3}$

FIGURE 2.28



Zeros of g: $x = \pm\sqrt{10}$

FIGURE 2.29



Zero of h: $t = \frac{3}{2}$

FIGURE 2.30

Zeros of a Function

If the graph of a function of x has an x-intercept at $(a, 0)$, then a is a **zero** of the function.

Zeros of a Function

The **zeros of a function** f of x are the x-values for which $f(x) = 0$.

Example 3 ▶ Finding the Zeros of a Function

Find the zeros of each function.

a. $f(x) = 3x^2 + x - 10$ **b.** $g(x) = \sqrt{10 - x^2}$ **c.** $h(t) = \dfrac{2t - 3}{t + 5}$

Solution

To find the zeros of a function, set the function equal to zero and solve for the independent variable.

a. $3x^2 + x - 10 = 0$ Set $f(x)$ equal to 0.

$(3x - 5)(x + 2) = 0$ Factor.

$3x - 5 = 0 \quad x = \frac{5}{3}$ Set 1st factor equal to 0.

$x + 2 = 0 \quad x = -2$ Set 2nd factor equal to 0.

The zeros of f are $x = \frac{5}{3}$ and $x = -2$. In Figure 2.28, note that the graph of f has $\left(\frac{5}{3}, 0\right)$ and $(-2, 0)$ as its x-intercepts.

b. $\sqrt{10 - x^2} = 0$ Set $g(x)$ equal to 0.

$10 - x^2 = 0$ Square each side.

$10 = x^2$ Add x^2 to each side.

$\pm\sqrt{10} = x$ Extract square root.

The zeros of g are $x = -\sqrt{10}$ and $x = \sqrt{10}$. In Figure 2.29, note that the graph of g has $(-\sqrt{10}, 0)$ and $(\sqrt{10}, 0)$ as its x-intercepts.

c. $\dfrac{2t - 3}{t + 5} = 0$ Set $h(t)$ equal to 0.

$2t - 3 = 0$ Multiply each side by $t + 5$.

$2t = 3$ Add 3 to each side.

$t = \dfrac{3}{2}$ Divide each side by 2.

The zero of h is $t = \frac{3}{2}$. In Figure 2.30, note that the graph of h has $\left(\frac{3}{2}, 0\right)$ as its t-intercept.

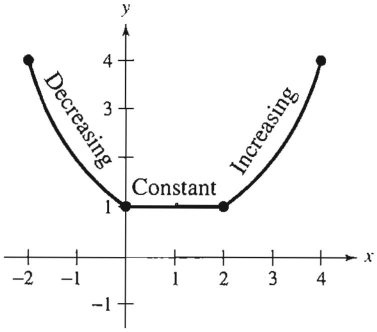


FIGURE 2.31

Increasing and Decreasing Functions

The more you know about the graph of a function, the more you know about the function itself. Consider the graph shown in Figure 2.31. As you move from *left to right*, this graph decreases, then is constant, and then increases.

Increasing, Decreasing, and Constant Functions

A function f is **increasing** on an interval if, for any x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

A function f is **decreasing** on an interval if, for any x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

A function f is **constant** on an interval if, for any x_1 and x_2 in the interval, $f(x_1) = f(x_2)$.

Example 4 ▶ Increasing and Decreasing Functions

In Figure 2.32, use the graphs to describe the increasing or decreasing behavior of each function.

Solution

a. This function is increasing over the entire real line.

b. This function is increasing on the interval $(-\infty, -1)$, decreasing on the interval $(-1, 1)$, and increasing on the interval $(1, \infty)$.

c. This function is increasing on the interval $(-\infty, 0)$, constant on the interval $(0, 2)$, and decreasing on the interval $(2, \infty)$.

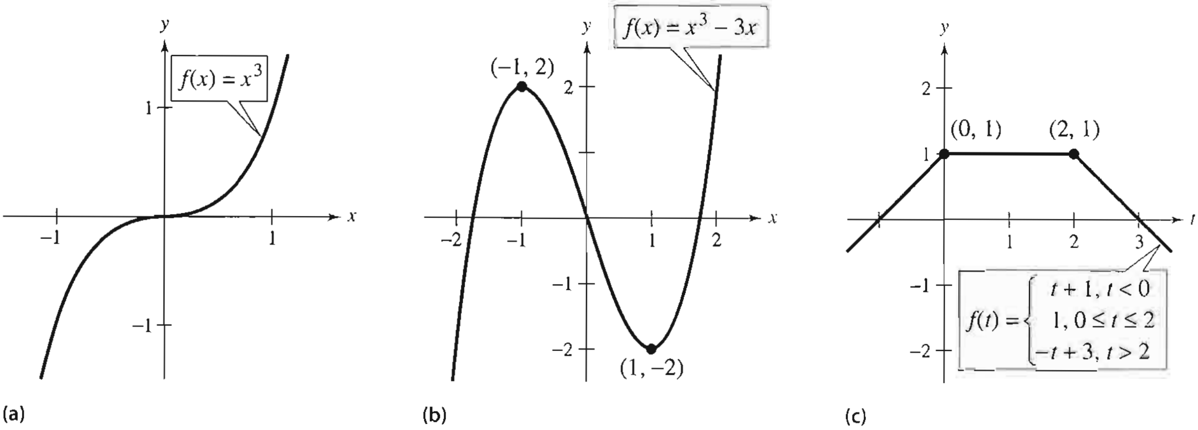


FIGURE 2.32

To help you decide whether a function is increasing, decreasing, or constant on an interval, you can evaluate the function for several values of x. However, calculus is needed to determine, for certain, all intervals on which a function is increasing, decreasing, or constant.

STUDY TIP

A relative minimum or relative maximum is also referred to as a *local* minimum or *local* maximum.

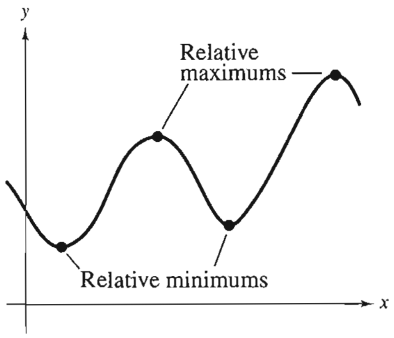


FIGURE 2.33

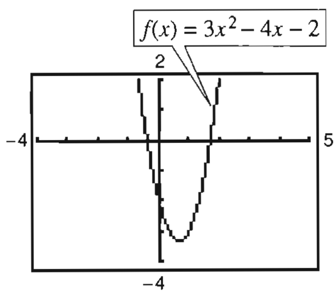


FIGURE 2.34

The points at which a function changes its increasing, decreasing, or constant behavior are helpful in determining the **relative minimum** or **relative maximum** values of the function.

Definition of Relative Minimum and Relative Maximum

A function value $f(a)$ is called a **relative minimum** of f if there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \le f(x).$$

A function value $f(a)$ is called a **relative maximum** of f if there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \ge f(x).$$

Figure 2.33 shows several different examples of relative minima and relative maxima. In Section 3.1, you will study a technique for finding the *exact point* at which a second-degree polynomial function has a relative minimum or relative maximum. For the time being, however, you can use a graphing utility to find reasonable approximations of these points.

Example 5 ▶ Approximating a Relative Minimum

Use a graphing utility to approximate the relative minimum of the function $f(x) = 3x^2 - 4x - 2$.

Solution

The graph of f is shown in Figure 2.34. By using the *zoom* and *trace* features of a graphing utility, you can estimate that the function has a relative minimum at the point

$$(0.67, -3.33). \qquad \text{Relative minimum}$$

Later, in Section 3.1, you will be able to determine that the exact point at which the relative minimum occurs is $\left(\frac{2}{3}, -\frac{10}{3}\right)$.

You can also use the *table* feature of a graphing utility to approximate numerically the relative minimum of the function in Example 5. Using a table that begins at 0.6 and increments the value of x by 0.01, you can approximate that the minimum of $f(x) = 3x^2 - 4x - 2$ occurs at the point $(0.67, -3.33)$.

Technology

If you use a graphing utility to estimate the x- and y-values of a relative minimum or relative maximum, the *automatic zoom* feature will often produce graphs that are nearly flat. To overcome this problem, you can manually change the vertical setting of the viewing window. The graph will stretch vertically if the values of Ymin and Ymax are closer together.

Even and Odd Functions

In Section 1.1, you studied different types of symmetry of a graph. In the terminology of functions, a function is said to be **even** if its graph is symmetric with respect to the y-axis and to be **odd** if its graph is symmetric with respect to the origin. The symmetry tests in Section 1.1 yield the following tests for even and odd functions.

Tests for Even and Odd Functions

A function $y = f(x)$ is **even** if, for each x in the domain of f,

$$f(-x) = f(x).$$

A function $y = f(x)$ is **odd** if, for each x in the domain of f,

$$f(-x) = -f(x).$$

Exploration

Graph each of the functions with a graphing utility. Determine whether the function is *even*, *odd*, or *neither*.

$f(x) = x^2 - x^4$

$g(x) = 2x^3 + 1$

$h(x) = x^5 - 2x^3 + x$

$j(x) = 2 - x^6 - x^8$

$k(x) = x^5 - 2x^4 + x - 2$

$p(x) = x^9 + 3x^5 - x^3 + x$

What do you notice about the equations of functions that are odd? What do you notice about the equations of functions that are even? Can you describe a way to identify a function as odd or even by inspecting the equation? Can you describe a way to identify a function as neither odd nor even by inspecting the equation?

Example 6 ▶ Even and Odd Functions

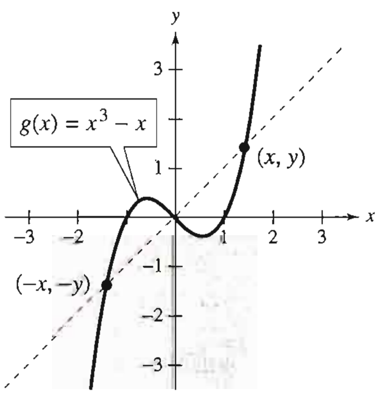
a. The function $g(x) = x^3 - x$ is odd because $g(-x) = -g(x)$, as follows.

$$\begin{aligned} g(-x) &= (-x)^3 - (-x) && \text{Substitute } -x \text{ for } x. \\ &= -x^3 + x && \text{Simplify.} \\ &= -(x^3 - x) && \text{Distributive Property} \\ &= -g(x) && \text{Test for odd function} \end{aligned}$$

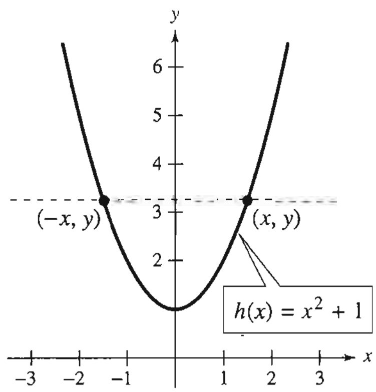
b. The function $h(x) = x^2 + 1$ is even because $h(-x) = h(x)$, as follows.

$$\begin{aligned} h(-x) &= (-x)^2 + 1 && \text{Substitute } -x \text{ for } x. \\ &= x^2 + 1 && \text{Simplify.} \\ &= h(x) && \text{Test for even function} \end{aligned}$$

The graphs of these two functions are shown in Figure 2.35.



(a) Symmetric to origin: Odd Function



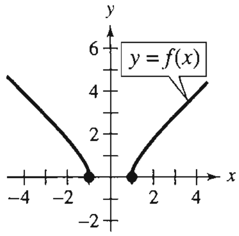
(b) Symmetric to y-axis: Even Function

FIGURE 2.35

2.3 Exercises

In Exercises 1–4, use the graph of the function to find the domain and range of *f*.

1.



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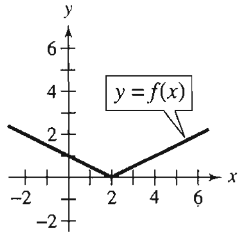

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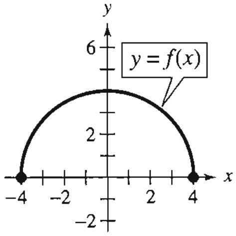
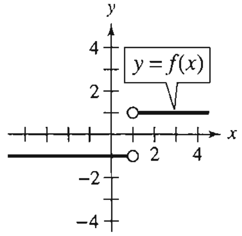
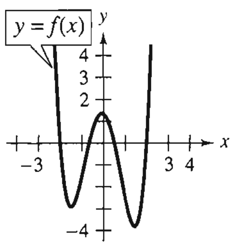


4.

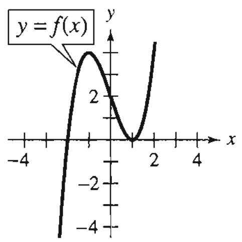


In Exercises 5–8, use the graph of the function to find the indicated function values.

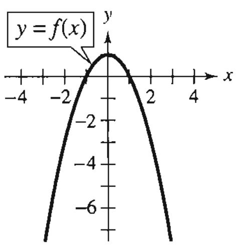
5. (a) $f(-2)$ (b) $f(-1)$ (c) $f(\frac{1}{2})$ (d) $f(1)$



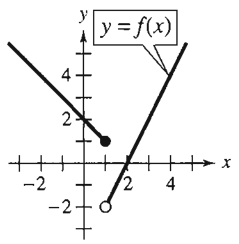
6. (a) $f(-1)$ (b) $f(2)$ (c) $f(0)$ (d) $f(1)$



7. (a) $f(-2)$ (b) $f(1)$ (c) $f(0)$ (d) $f(2)$

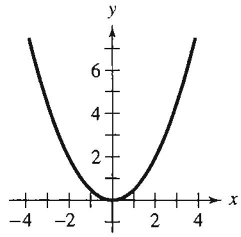


8. (a) $f(2)$ (b) $f(1)$ (c) $f(3)$ (d) $f(-1)$

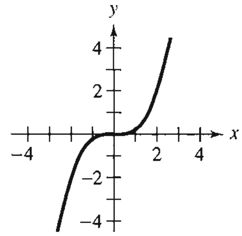


In Exercises 9–14, use the Vertical Line Test to determine whether *y* is a function of *x*. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

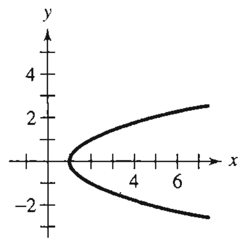
9. $y = \frac{1}{2}x^2$



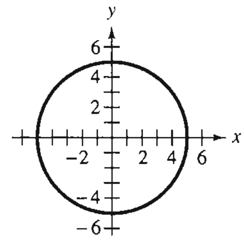
10. $y = \frac{1}{4}x^3$



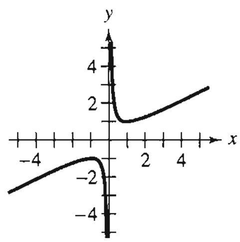
11. $x - y^2 = 1$



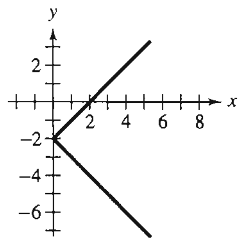
12. $x^2 + y^2 = 25$



13. $x^2 = 2xy - 1$



14. $x = |y + 2|$



In Exercises 15–24, find the zeros of the function algebraically.

15. $f(x) = 2x^2 - 7x - 30$

16. $f(x) = 3x^2 + 22x - 16$

17. $f(x) = \dfrac{x}{9x^2 - 4}$

18. $f(x) = \dfrac{x^2 - 9x + 14}{4x}$

19. $f(x) = \frac{1}{2}x^3 - x$

20. $f(x) = x^3 - 4x^2 - 9x + 36$

21. $f(x) = 4x^3 - 24x^2 - x + 6$

22. $f(x) = 9x^4 - 25x^2$

23. $f(x) = \sqrt{2x} - 1$

24. $f(x) = \sqrt{3x + 2}$

In Exercises 25–30, use a graphing utility to graph the function and find the zeros of the function. Verify your results algebraically.

25. $f(x) = 3 + \dfrac{5}{x}$

26. $f(x) = x(x - 7)$

27. $f(x) = \sqrt{2x + 11}$

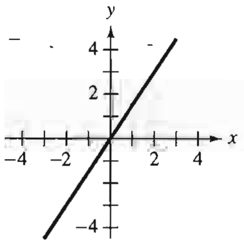
28. $f(x) = \sqrt{3x - 14} - 8$

29. $f(x) = \dfrac{3x - 1}{x - 6}$

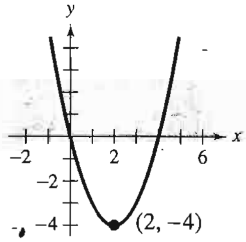
30. $f(x) = \dfrac{2x^2 - 9}{3 - x}$

In Exercises 31–38, determine the intervals over which the function is increasing, decreasing, or constant.

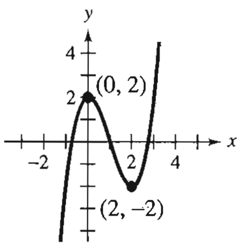
31. $f(x) = \frac{3}{2}x$



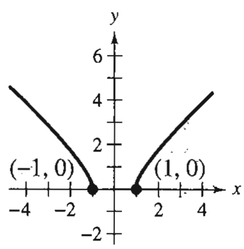
32. $f(x) = x^2 - 4x$



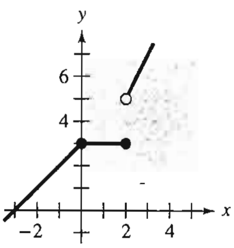
33. $f(x) = x^3 - 3x^2 + 2$



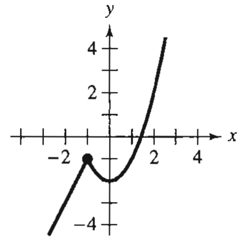
34. $f(x) = \sqrt{x^2 - 1}$



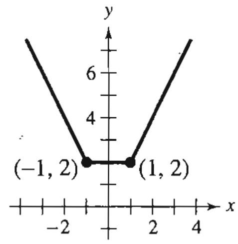
35. $f(x) = \begin{cases} x + 3, & x \le 0 \\ 3, & 0 < x \le 2 \\ 2x + 1, & x > 2 \end{cases}$



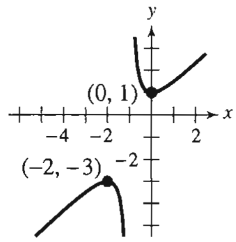
36. $f(x) = \begin{cases} 2x + 1, & x \le -1 \\ x^2 - 2, & x > -1 \end{cases}$



37. $f(x) = |x + 1| + |x - 1|$



38. $f(x) = \dfrac{x^2 + x + 1}{x + 1}$



In Exercises 39–48, (a) use a graphing utility to graph the function and visually determine the intervals over which the function is increasing, decreasing, or constant, and (b) make a table of values to verify whether the function is increasing, decreasing, or constant over the intervals you identified in part (a).

39. $f(x) = 3$

40. $g(x) = x$

41. $g(s) = \dfrac{s^2}{4}$

42. $h(x) = x^2 - 4$

43. $f(t) = -t^4$

44. $f(x) = 3x^4 - 6x^2$

45. $f(x) = \sqrt{1 - x}$

46. $f(x) = x\sqrt{x + 3}$

47. $f(x) = x^{3/2}$

48. $f(x) = x^{2/3}$

In Exercises 49–52, use a graphing utility to approximate the relative minimum/relative maximum of each function.

49. $f(x) = (x - 4)(x + 2)$

50. $f(x) = 3x^2 - 2x - 5$

51. $f(x) = x(x - 2)(x + 3)$

52. $f(x) = x^3 - 3x^2 - x + 1$

In Exercises 53–60, graph the function and determine the interval(s) for which $f(x) \ge 0$.

53. $f(x) = 4 - x$

54. $f(x) = 4x + 2$

55. $f(x) = x^2 + x$

56. $f(x) = x^2 - 4x$

57. $f(x) = \sqrt{x - 1}$

58. $f(x) = \sqrt{x + 2}$

59. $f(x) = -(1 + |x|)$

60. $f(x) = \frac{1}{2}(2 + |x|)$

In Exercises 61–66, determine whether the function is even, odd, or neither.

61. $f(x) = x^6 - 2x^2 + 3$

62. $h(x) = x^3 - 5$

63. $g(x) = x^3 - 5x$

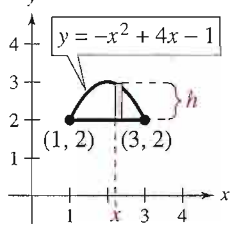
64. $f(x) = x\sqrt{1 - x^2}$

65. $f(t) = t^2 + 2t - 3$

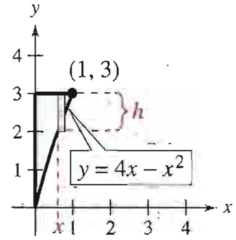
66. $g(s) = 4s^{2/3}$

In Exercises 67–70, write the height h of the rectangle as a function of x.

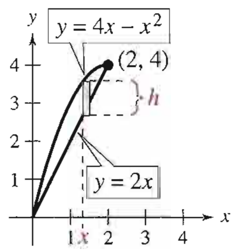
67.



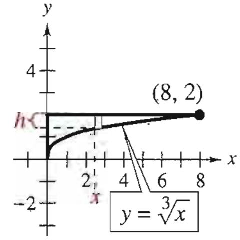
68.



69.

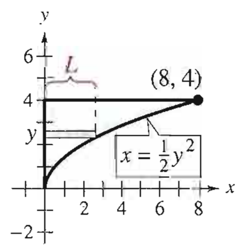


70.

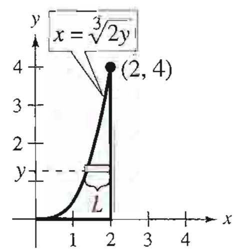


In Exercises 71–74, write the length L of the rectangle as a function of y.

71.



72.



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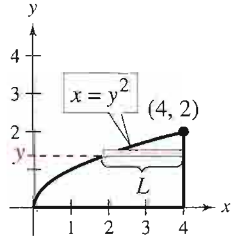
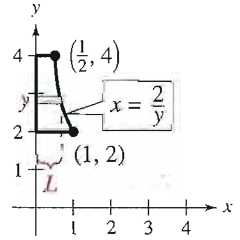


74.



75. ***Electronics*** The number of lumens (time rate of flow of light) L from a fluorescent lamp can be approximated by the model

$$L = -0.294x^2 + 97.744x - 664.875, \quad 20 \le x \le 90$$

where x is the wattage of the lamp.

(a) Use a graphing utility to graph the function.

(b) Use the graph from part (a) to estimate the wattage necessary to obtain 2000 lumens.

▶ Model It

76. ***Data Analysis*** The table shows the amount y (in billions of dollars) of the merchandise trade balance of the United States for the years 1991 through 1999. The merchandise trade balance is the difference between the values of exports and imports. A negative merchandise trade balance indicates that imports exceeded exports. (Source: U.S. International Trade Administration and U.S. Foreign Trade Highlights)

Year, x	Trade balance, y
1991	−66.8
1992	−84.5
1993	−115.6
1994	−150.7
1995	−158.7
1996	−170.2
1997	−181.5
1998	−229.8
1999	−330.0

(a) Use a graphing utility to create a scatter plot of the data.

(b) Use the graph in part (a) to determine whether the data represents y as a function of x.

(c) Use the *regression* feature of a graphing utility to find a cubic model (a model of the form $y = ax^3 + bx^2 + cx + d$) for the data. Let x be the time (in years), with $x = 1$ corresponding to 1991.

(d) What is the domain of the model?

(e) Use a graphing utility to graph the model in the same viewing window you used in part (a).

(f) For which year does the model most accurately estimate the actual data? During which year is it least accurate?

77. ***Coordinate Axis Scale*** Each function models the specified data for the years 1995 through 2002, with $t = 5$ corresponding to 1995. Estimate a reasonable scale for the vertical axis (e.g., hundreds, thousands, millions, etc.) of the graph and justify your answer. (There are many correct answers.)

(a) $f(t)$ represents the average salary of college professors.

(b) $f(t)$ represents the U.S. population.

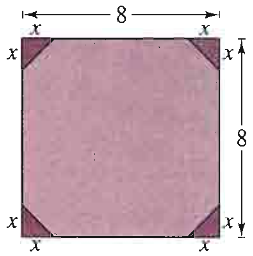
(c) $f(t)$ represents the percent of the civilian work force that is unemployed.

78. ***Geometry*** Corners of equal size are cut from a square with sides of length 8 meters (see figure).

(a) Write the area A of the resulting figure as a function of x. Determine the domain of the function.

(b) Use a graphing utility to graph the area function over its domain. Use the graph to find the range of the function.

(c) Identify the figure that would result if x were chosen to be the maximum value in the domain of the function. What would be the length of each side of the figure?



Synthesis

True or False? **In Exercises 79 and 80, determine whether the statement is true or false. Justify your answer.**

79. A function with a square root cannot have a domain that is the set of real numbers.

80. It is possible for an odd function to have the interval $[0, \infty)$ as its domain.

81. If f is an even function, determine whether g is even, odd, or neither. Explain.

(a) $g(x) = -f(x)$ (b) $g(x) = f(-x)$

(c) $g(x) = f(x) - 2$ (d) $g(x) = f(x - 2)$

82. ***Think About It*** Does the graph in Exercise 11 represent x as a function of y? Explain.

Think About It **In Exercises 83–86, find the coordinates of a second point on the graph of a function f if the given point is on the graph and the function is (a) even and (b) odd.**

83. $\left(-\frac{3}{2}, 4\right)$ **84.** $\left(-\frac{5}{3}, -7\right)$

85. $(4, 9)$ **86.** $(5, -1)$

87. ***Writing*** Use a graphing utility to graph each function. Write a paragraph describing any similarities and differences you observe among the graphs.

(a) $y = x$ (b) $y = x^2$

(c) $y = x^3$ (d) $y = x^4$

(e) $y = x^5$ (f) $y = x^6$

88. ***Conjecture*** Use the results of Exercise 87 to make a conjecture about the graphs of $y = x^7$ and $y = x^8$. Use a graphing utility to graph the functions and compare the results with your conjecture.

Review

In Exercises 89–92, solve the equation.

89. $x^2 - 10x = 0$ **90.** $100 - (x - 5)^2 = 0$

91. $x^3 - x = 0$ **92.** $16x^2 - 40x + 25 = 0$

In Exercises 93–96, evaluate the function at each specified value of the independent variable and simplify.

93. $f(x) = 5x - 8$

(a) $f(9)$ (b) $f(-4)$ (c) $f(x - 7)$

94. $f(x) = x^2 - 10x$

(a) $f(4)$ (b) $f(-8)$ (c) $f(x - 4)$

95. $f(x) = \sqrt{x - 12} - 9$

(a) $f(12)$ (b) $f(40)$ (c) $f\left(-\sqrt{36}\right)$

96. $f(x) = x^4 - x - 5$

(a) $f(-1)$ (b) $f\left(\frac{1}{2}\right)$ (c) $f\left(2\sqrt{3}\right)$

In Exercises 97 and 98, find the difference quotient and simplify your answer.

97. $f(x) = x^2 - 2x + 9, \quad \dfrac{f(3 + h) - f(3)}{h}, \quad h \neq 0$

98. $f(x) = 5 + 6x - x^2, \quad \dfrac{f(6 + h) - f(6)}{h}, \quad h \neq 0$

2.4 A Library of Functions

▶ What you should learn

- How to identify and graph linear and squaring functions
- How to identify and graph cubic, square root, and reciprocal functions
- How to identify and graph step and other piecewise-defined functions
- How to recognize graphs of common functions

▶ Why you should learn it

Piecewise-defined functions can be used to model real-life situations. For instance, in Exercise 68 on page 218, you will use a piecewise-defined function to model the monthly revenue of a landscaping business.

Michael Newman/PhotoEdit

Linear and Squaring Functions

One of the goals of this text is to enable you to recognize the basic shapes of the graphs of different types of functions. For instance, you know that the graph of the **linear function** $f(x) = ax + b$ is a line with slope $m = a$ and y-intercept at $(0, b)$. The graph of the linear function has the following features.

- The domain of the function is the set of all real numbers.
- The range of the function is the set of all real numbers.
- The graph has one intercept, $(0, b)$.
- The graph is increasing if $m > 0$, decreasing if $m < 0$, and constant if $m = 0$.

Example 1 ▶ Writing a Linear Function

Write the linear function f for which $f(1) = 3$ and $f(4) = 0$.

Solution

To find the equation of the line that passes through $(x_1, y_1) = (1, 3)$ and $(x_2, y_2) = (4, 0)$, first find the slope of the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 3}{4 - 1} = \frac{-3}{3} = -1$$

Next, use the point-slope form of the equation of a line.

$y - y_1 = m(x - x_1)$	Point-slope form
$y - 3 = -1(x - 1)$	Substitute for x_1, y_1, and m.
$y = -x + 4$	Simplify.
$f(x) = -x + 4$	Function notation

The graph of this function is shown in Figure 2.36.

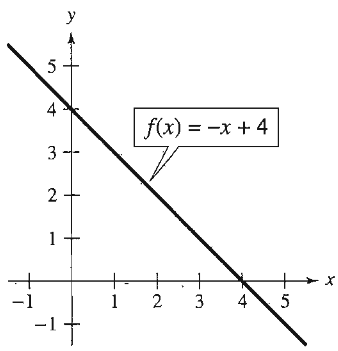


FIGURE 2.36

There are two special types of linear functions, the **constant function** and the **identity function.** A constant function has the form

$$f(x) = c$$

and has the domain of all real numbers with a range consisting of a single real number c. The graph of a constant function is a horizontal line, as shown in Figure 2.37. The identity function has the form

$$f(x) = x.$$

Its domain and range are the set of all real numbers. The identity function has a slope of $m = 1$ and a y-intercept $(0, 0)$. The graph of the identity function is a line for which each x-coordinate equals the corresponding y-coordinate. The graph is always increasing, as shown in Figure 2.38.

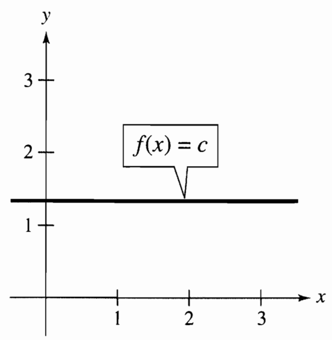


FIGURE 2.37

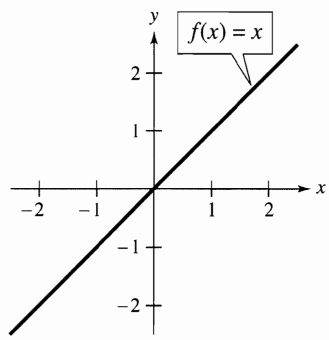


FIGURE 2.38

The graph of the **squaring function**

$$f(x) = x^2$$

is a U-shaped curve with the following features.

- The domain of the function is the set of all real numbers.
- The range of the function is the set of all nonnegative real numbers.
- The function is even.
- The graph has an intercept at $(0, 0)$.
- The graph is decreasing on the interval $(-\infty, 0)$ and increasing on the interval $(0, \infty)$.
- The graph is symmetric with respect to the y-axis.
- The graph has a relative minimum at $(0, 0)$.

The graph of the squaring function is shown in Figure 2.39.

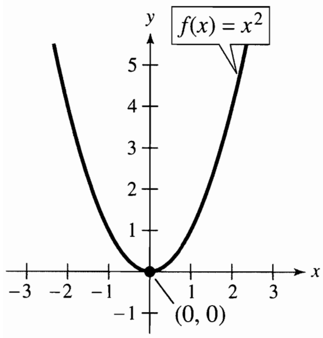


FIGURE 2.39

Cubic, Square Root, and Reciprocal Functions

Special features of the graphs of the **cubic, square root,** and **reciprocal functions** are summarized below.

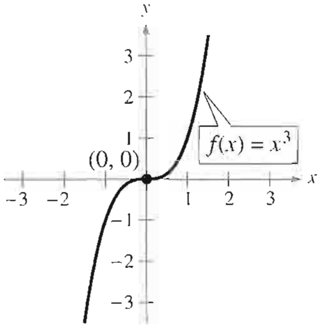
1. The graph of the *cubic* function

$$f(x) = x^3$$

has the following features.

- The domain of the function is the set of all real numbers.
- The range of the function is the set of all real numbers.
- The function is odd.
- The graph has an intercept at $(0, 0)$.
- The graph is increasing on the interval $(-\infty, \infty)$.
- The graph is symmetric with respect to the origin.

The graph of the cubic function is shown in Figure 2.40.



Cubic function
FIGURE 2.40

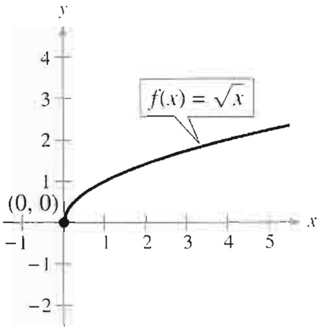
2. The graph of the *square root* function

$$f(x) = \sqrt{x}$$

has the following features.

- The domain of the function is the set of all nonnegative real numbers.
- The range of the function is the set of all nonnegative real numbers.
- The graph has an intercept at $(0, 0)$.
- The graph is increasing on the interval $(0, \infty)$.

The graph of the square root function is shown in Figure 2.41.



Square root function
FIGURE 2.41

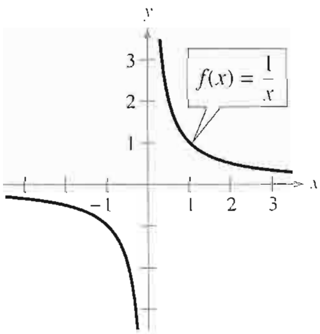
3. The graph of the *reciprocal* function

$$f(x) = \frac{1}{x}$$

has the following features.

- The domain of the function is $(-\infty, 0) \cup (0, \infty)$.
- The range of the function is $(-\infty, 0) \cup (0, \infty)$.
- The function is odd.
- The graph does not have any intercepts.
- The graph is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$.
- The graph is symmetric with respect to the origin.

The graph of the reciprocal function is shown in Figure 2.42.



Reciprocal function
FIGURE 2.42

Step and Piecewise-Defined Functions

Functions whose graphs resemble sets of stairsteps are known as **step functions.** The most famous of the step functions is the **greatest integer function,** which is denoted by $[\![x]\!]$ and defined as

$$f(x) = [\![x]\!] = \textit{the greatest integer less than or equal to } x.$$

Some values of the greatest integer function are as follows.

$$[\![-1]\!] = (\text{greatest integer} \le -1) = -1$$

$$[\![-\tfrac{1}{2}]\!] = (\text{greatest integer} \le -\tfrac{1}{2}) = -1$$

$$[\![\tfrac{1}{10}]\!] = (\text{greatest integer} \le \tfrac{1}{10}) = 0$$

$$[\![1.5]\!] = (\text{greatest integer} \le 1.5) = 1$$

The graph of the greatest integer function

$$f(x) = [\![x]\!]$$

has the following features, as shown in Figure 2.43.

- The domain of the function is the set of all real numbers.
- The range of the function is the set of all integers.
- The graph has a y-intercept at $(0, 0)$ and x-intercepts in the interval $[0, 1)$.
- The graph is constant between each pair of consecutive integers.
- The graph jumps vertically one unit at each integer value.

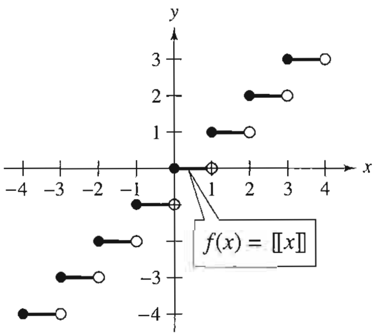


FIGURE 2.43

Technology

When graphing a step function, you should set your graphing utility to *dot* mode.

Example 2 ▶ Evaluating a Step Function

Evaluate the function when $x = -1$, 2, and $\frac{3}{2}$.

$$f(x) = [\![x]\!] + 1$$

Solution

For $x = -1$, the greatest integer ≤ -1 is -1, so

$$f(-1) = [\![-1]\!] + 1 = -1 + 1 = 0.$$

For $x = 2$, the greatest integer ≤ 2 is 2, so

$$f(2) = [\![2]\!] + 1 = 2 + 1 = 3.$$

For $x = \frac{3}{2}$, the greatest integer $\le \frac{3}{2}$ is 1, so

$$f(\tfrac{3}{2}) = [\![\tfrac{3}{2}]\!] + 1 = 1 + 1 = 2.$$

The graph of $f(x) = [\![x]\!] + 1$ is shown in Figure 2.44.

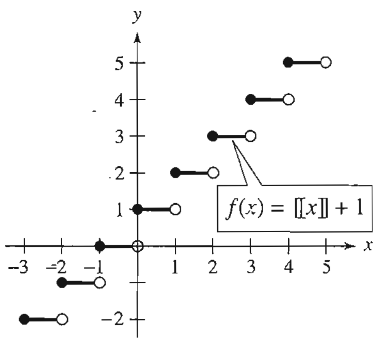


FIGURE 2.44

Recall from Section 2.2 that a piecewise-defined function is defined by two or more equations over a specified domain. To graph a piecewise-defined function, graph each equation separately over the specified domain, as shown in Example 3.

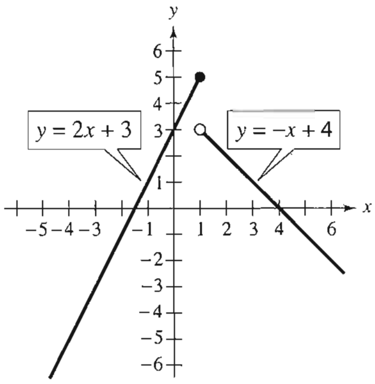


FIGURE 2.45

Example 3 ▶ Graphing a Piecewise-Defined Function

Sketch the graph of

$$f(x) = \begin{cases} 2x + 3, & x \le 1 \\ -x + 4, & x > 1 \end{cases}.$$

Solution

This piecewise-defined function is composed of two linear functions. At $x = 1$ and to the left of $x = 1$ the graph is the line $y = 2x + 3$, and to the right of $x = 1$ the graph is the line $y = -x + 4$, as shown in Figure 2.45.

Common Functions

The eight graphs shown in Figure 2.46 represent the most commonly used functions in algebra. Familiarity with the basic characteristics of these simple graphs will help you analyze the shapes of more complicated graphs—in particular, graphs obtained from these graphs by the rigid and nonrigid transformations studied in the next section.

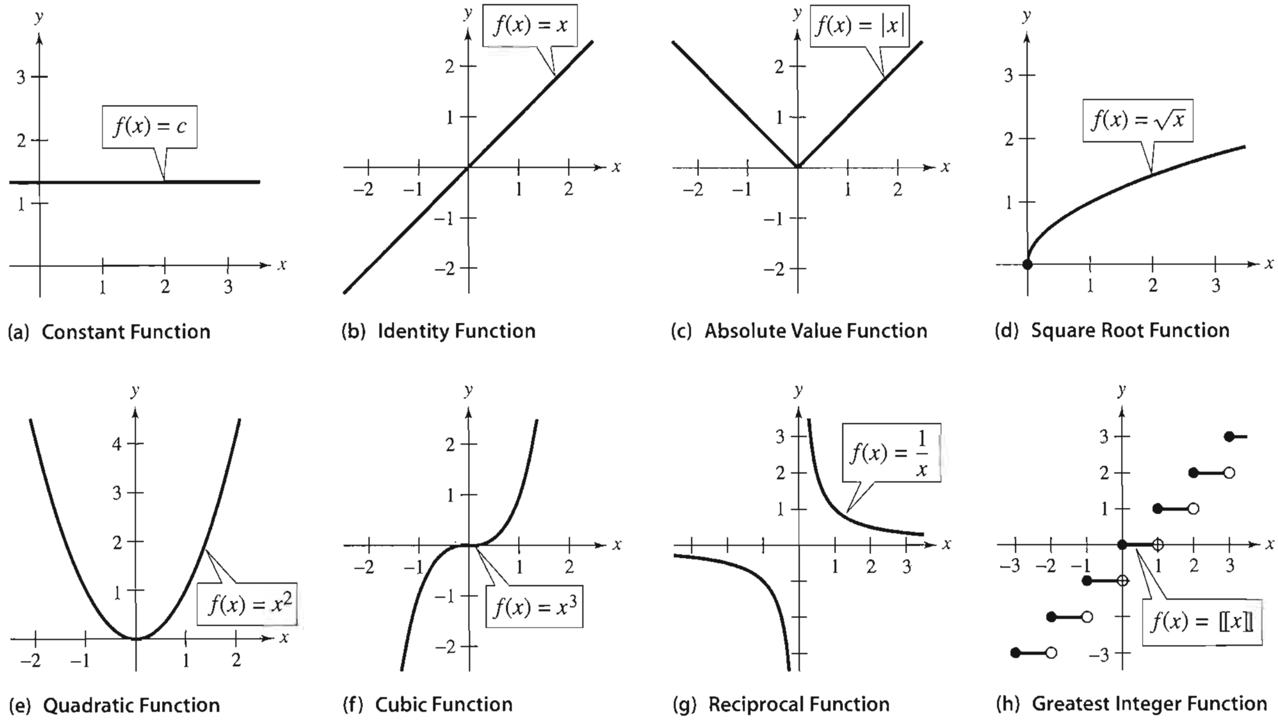


FIGURE 2.46

2.4 Exercises

In Exercises 1–8, write the linear function that has the indicated function values. Then sketch the graph of the function.

1. $f(1) = 4, f(0) = 6$
2. $f(-3) = -8, f(1) = 2$
3. $f(5) = -4, f(-2) = 17$
4. $f(3) = 9, f(-1) = -11$
5. $f(-5) = -1, f(5) = -1$
6. $f(-10) = 12, f(16) = -1$
7. $f\left(\frac{1}{2}\right) = -6, f(4) = -3$
8. $f\left(\frac{2}{3}\right) = -\frac{15}{2}, f(-4) = -11$

In Exercises 9–28, use a graphing utility to graph the function. Be sure to choose an appropriate viewing window.

9. $f(x) = -x - \frac{3}{4}$
10. $f(x) = 3x - \frac{5}{2}$
11. $f(x) = -\frac{1}{6}x - \frac{5}{2}$
12. $f(x) = \frac{5}{6} - \frac{2}{3}x$
13. $f(x) = x^2 - 2x$
14. $f(x) = -x^2 + 8x$
15. $h(x) = -x^2 + 4x + 12$
16. $g(x) = x^2 - 6x - 16$
17. $f(x) = x^3 - 1$
18. $f(x) = 8 - x^3$
19. $f(x) = (x - 1)^3 + 2$
20. $g(x) = 2(x + 3)^3 + 1$
21. $f(x) = 4\sqrt{x}$
22. $f(x) = 4 - 2\sqrt{x}$
23. $g(x) = 2 - \sqrt{x + 4}$
24. $h(x) = \sqrt{x + 2} + 3$
25. $f(x) = -\dfrac{1}{x}$
26. $f(x) = 4 + \dfrac{1}{x}$
27. $h(x) = \dfrac{1}{x + 2}$
28. $k(x) = \dfrac{1}{x - 3}$

In Exercise 29–36, evaluate the function for the indicated values.

29. $f(x) = [\![x]\!]$
    (a) $f(2.1)$ (b) $f(2.9)$ (c) $f(-3.1)$ (d) $f\left(\frac{7}{2}\right)$
30. $g(x) = 2[\![x]\!]$
    (a) $g(-3)$ (b) $g(0.25)$ (c) $g(9.5)$ (d) $g\left(\frac{11}{3}\right)$
31. $h(x) = [\![x + 3]\!]$
    (a) $h(-2)$ (b) $h\left(\frac{1}{2}\right)$ (c) $h(4.2)$ (d) $h(-21.6)$
32. $f(x) = 4[\![x]\!] + 7$
    (a) $f(0)$ (b) $f(-1.5)$ (c) $f(6)$ (d) $f\left(\frac{5}{3}\right)$
33. $h(x) = [\![3x - 1]\!]$
    (a) $h(2.5)$ (b) $h(-3.2)$ (c) $h\left(\frac{7}{3}\right)$ (d) $h\left(-\frac{21}{3}\right)$
34. $k(x) = [\![\frac{1}{2}x + 6]\!]$
    (a) $k(5)$ (b) $k(-6.1)$ (c) $k(0.1)$ (d) $k(15)$
35. $g(x) = 3[\![x - 2]\!] + 5$
    (a) $g(-2.7)$ (b) $g(-1)$ (c) $g(0.8)$ (d) $g(14.5)$
36. $g(x) = -7[\![x + 4]\!] + 6$
    (a) $g\left(\frac{1}{8}\right)$ (b) $g(9)$ (c) $g(-4)$ (d) $g\left(\frac{3}{2}\right)$

In Exercises 37–42, sketch the graph of the function.

37. $g(x) = -[\![x]\!]$
38. $g(x) = 4[\![x]\!]$
39. $g(x) = [\![x]\!] - 2$
40. $g(x) = [\![x]\!] - 1$
41. $g(x) = [\![x + 1]\!]$
42. $g(x) = [\![x - 3]\!]$

In Exercises 43–50, graph the function.

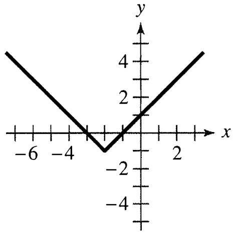
43. $f(x) = \begin{cases} 2x + 3, & x < 0 \\ 3 - x, & x \geq 0 \end{cases}$
44. $g(x) = \begin{cases} x + 6, & x \leq -4 \\ \frac{1}{2}x - 4, & x > -4 \end{cases}$
45. $f(x) = \begin{cases} \sqrt{4 + x}, & x < 0 \\ \sqrt{4 - x}, & x \geq 0 \end{cases}$
46. $f(x) = \begin{cases} 1 - (x - 1)^2, & x \leq 2 \\ \sqrt{x - 2}, & x > 2 \end{cases}$
47. $f(x) = \begin{cases} x^2 + 5, & x \leq 1 \\ -x^2 + 4x + 3, & x > 1 \end{cases}$
48. $h(x) = \begin{cases} 3 - x^2, & x < 0 \\ x^2 + 2, & x \geq 0 \end{cases}$
49. $h(x) = \begin{cases} 4 - x^2, & x < -2 \\ 3 + x, & -2 \leq x < 0 \\ x^2 + 1, & x \geq 0 \end{cases}$
50. $k(x) = \begin{cases} 2x + 1, & x \leq -1 \\ 2x^2 - 1, & -1 < x \leq 1 \\ 1 - x^2, & x > 1 \end{cases}$

In Exercises 51 and 52, use a graphing utility to graph the function. State the domain and range of the function. Describe the pattern of the graph.

51. $s(x) = 2\left(\frac{1}{4}x - [\![\frac{1}{4}x]\!]\right)$
52. $g(x) = 2\left(\frac{1}{4}x - [\![\frac{1}{4}x]\!]\right)^2$

In Exercises 53–62, identify the common function and the transformed common function shown in the graph. Write an equation for the function shown in the graph. Then use a graphing utility to verify your answer.

53.



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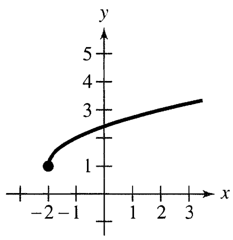

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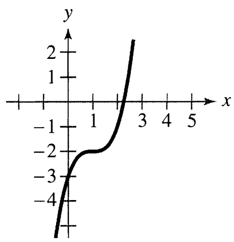

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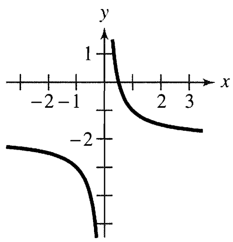

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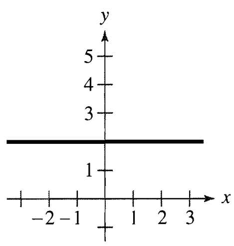

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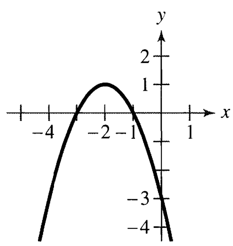

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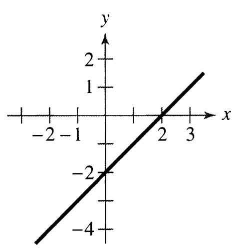

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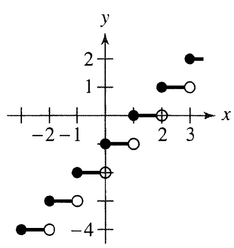

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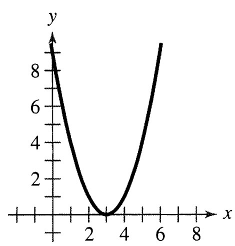
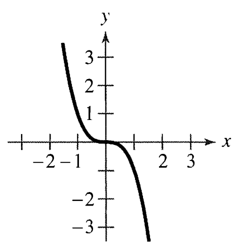


62.



63. ***Communications*** The cost of a telephone call between Denver and Boise is \$0.60 for the first minute and \$0.42 for each additional minute or portion of a minute. A model for the total cost C (in dollars) of the phone call is

$$C = 0.60 - 0.42[\![1 - t]\!], \quad t > 0$$

where t is the length of the phone call in minutes.

(a) Sketch the graph of the model.

(b) Determine the cost of a call lasting 12 minutes and 30 seconds.

64. ***Communications*** The cost of using a telephone calling card is \$1.05 for the first minute and \$0.38 for each additional minute or portion of a minute.

(a) A customer needs a model for the cost C of using a calling card for a call lasting t minutes. Which of the following is the appropriate model? Explain.

$$C_1(t) = 1.05 + 0.38[\![t - 1]\!]$$

$$C_2(t) = 1.05 - 0.38[\![-(t - 1)]\!]$$

(b) Graph the appropriate model. Determine the cost of a call lasting 18 minutes and 45 seconds.

65. ***Delivery Charges*** The cost of sending an overnight package from Los Angeles to Miami is \$10.75 for a package weighing up to but not including 1 pound and \$3.95 for each additional pound or portion of a pound. A model for the total cost C (in dollars) of sending the package is

$$C = 10.75 + 3.95[\![x]\!], \quad x > 0$$

where x is the weight in pounds.

(a) Sketch a graph of the model.

(b) Determine the cost of sending a package that weighs 10.33 pounds.

66. ***Delivery Charges*** The cost of sending an overnight package from New York to Atlanta is \$9.80 for a package weighing up to but not including 1 pound and \$2.50 for each additional pound or portion of a pound.

(a) Use the greatest integer function to create a model for the cost C of overnight delivery of a package weighing x pounds, $x > 0$.

(b) Sketch the graph of the function.

67. ***Wages*** A mechanic is paid \$12.00 per hour for regular time and time-and-a-half for overtime. The weekly wage function is

$$W(h) = \begin{cases} 12h, & 0 < h \le 40 \\ 18(h - 40) + 480, & h > 40 \end{cases}$$

where h is the number of hours worked in a week.

(a) Evaluate $W(30)$, $W(40)$, $W(45)$, and $W(50)$.

(b) The company increased the regular work week to 45 hours. What is the new weekly wage function?

▶ Model It

68. ***Revenue*** The table shows the monthly revenue y (in thousands of dollars) of a landscaping business for the year 2002, with $x = 1$ representing January.

Month, x	Revenue, y
1	5.2
2	5.6
3	6.6
4	8.3
5	11.5
6	15.8
7	12.8
8	10.1
9	8.6
10	6.9
11	4.5
12	2.7

A mathematical model that represents this data is

$$f(x) = \begin{cases} -1.97x + 26.3 \\ 0.505x^2 - 1.47x + 6.3 \end{cases}.$$

(a) What is the domain of each part of the piecewise-defined function? How can you tell? Explain your reasoning.

(b) Sketch a graph of the model.

(c) Find $f(5)$ and $f(11)$, and interpret your results in the context of the problem.

(d) How do the values obtained from the model in part (b) compare with the actual data values?

69. ***Fluid Flow*** The intake pipe of a 100-gallon tank has a flow rate of 10 gallons per minute, and two drainpipes have flow rates of 5 gallons per minute each. The figure shows the volume V of fluid in the tank as a function of time t. Determine the combination of the input pipe and drain pipes in which the fluid is flowing in specific subintervals of the 1 hour of time shown on the graph. (There are many correct answers.)

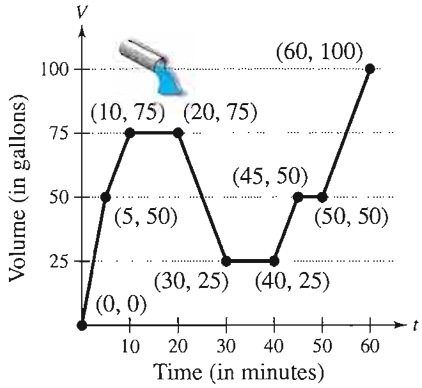


FIGURE FOR 69

Synthesis

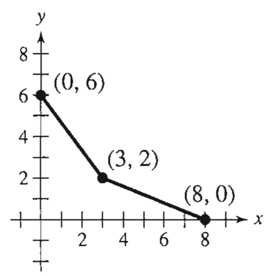
True or False? **In Exercises 70 and 71, determine whether the statement is true or false. Justify your answer.**

70. A piecewise-defined function will always have at least one x-intercept or at least one y-intercept.

71. $f(x) = \begin{cases} 2, & 1 \le x < 2 \\ 4, & 2 \le x < 3 \\ 6, & 3 \le x < 4 \end{cases}$

can be rewritten as $f(x) = 2[\![x]\!]$, $1 \le x < 4$.

72. ***Exploration*** Write equations for the piecewise-defined function shown in the graph.



Review

In Exercise 73 and 74, solve the inequality and sketch the solution on the real number line.

73. $3x + 4 \le 12 - 5x$

74. $2x + 1 > 6x - 9$

In Exercises 75 and 76, determine whether the lines L_1 and L_2 passing through the pairs of points are parallel, perpendicular, or neither.

75. L_1: $(-2, -2), (2, 10)$
L_2: $(-1, 3), (3, 9)$

76. L_1: $(-1, -7), (4, 3)$
L_2: $(1, 5), (-2, -7)$

2.5 Shifting, Reflecting, and Stretching Graphs

▶ What you should learn

- How to use vertical and horizontal shifts to sketch graphs of functions
- How to use reflections to sketch graphs of functions
- How to use nonrigid transformations to sketch graphs of functions

▶ Why you should learn it

Knowing the graphs of common functions and knowing how to shift, reflect, and stretch graphs of functions can help you sketch a wide variety of simple functions by hand. This skill is useful in sketching graphs of functions that model real-life data, such as in Exercise 63 on page 227, where you are asked to sketch the graph of a function that models the amount of fuel used by trucks from 1980 through 1999.

Chuck Keeler/The Stock Market

Shifting Graphs

Many functions have graphs that are simple transformations of the common graphs summarized in Section 2.4. For example, you can obtain the graph of

$$h(x) = x^2 + 2$$

by shifting the graph of $f(x) = x^2$ *up* two units, as shown in Figure 2.47. In function notation, h and f are related as follows.

$$h(x) = x^2 + 2 = f(x) + 2 \qquad \text{Upward shift of two units}$$

Similarly, you can obtain the graph of

$$g(x) = (x - 2)^2$$

by shifting the graph of $f(x) = x^2$ to the *right* two units, as shown in Figure 2.48. In this case, the functions g and f have the following relationship.

$$g(x) = (x - 2)^2 = f(x - 2) \qquad \text{Right shift of two units}$$

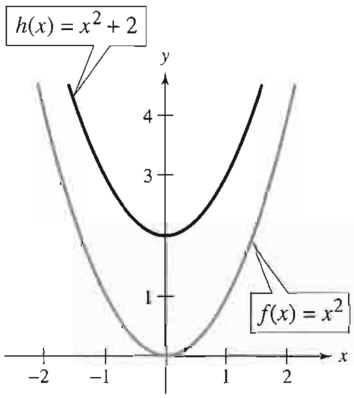


FIGURE 2.47

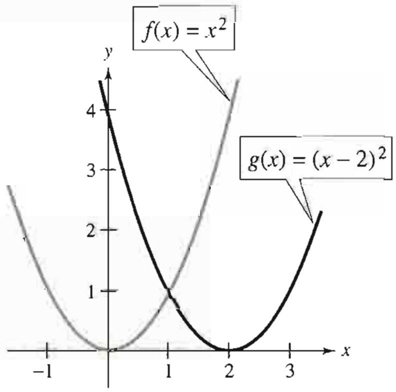


FIGURE 2.48

The following list summarizes this discussion about horizontal and vertical shifts.

Vertical and Horizontal Shifts

Let c be a positive real number. **Vertical and horizontal shifts** in the graph of $y = f(x)$ are represented as follows.

1. Vertical shift c units *upward:* $\quad h(x) = f(x) + c$
2. Vertical shift c units *downward:* $\quad h(x) = f(x) - c$
3. Horizontal shift c units to the *right:* $\quad h(x) = f(x - c)$
4. Horizontal shift c units to the *left:* $\quad h(x) = f(x + c)$

STUDY TIP

In items 3 and 4, be sure you see that $h(x) = f(x - c)$ corresponds to a *right* shift and $h(x) = f(x + c)$ corresponds to a *left* shift for $c > 0$.

Some graphs can be obtained from combinations of vertical and horizontal shifts, as demonstrated in Example 1(b). Vertical and horizontal shifts generate a *family of functions*, each with the same shape but at different locations in the plane.

Example 1 ▶ Shifts in the Graphs of a Function

Use the graph of $f(x) = x^3$ to sketch the graph of each function.

a. $g(x) = x^3 - 1$

b. $h(x) = (x + 2)^3 + 1$

Solution

a. Relative to the graph of $f(x) = x^3$, the graph of $g(x) = x^3 - 1$ is a downward shift of one unit, as shown in Figure 2.49.

b. Relative to the graph of $f(x) = x^3$, the graph of $h(x) = (x + 2)^3 + 1$ involves a left shift of two units and an upward shift of one unit, as shown in Figure 2.50.

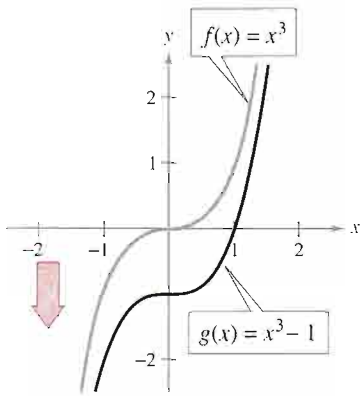


FIGURE 2.49

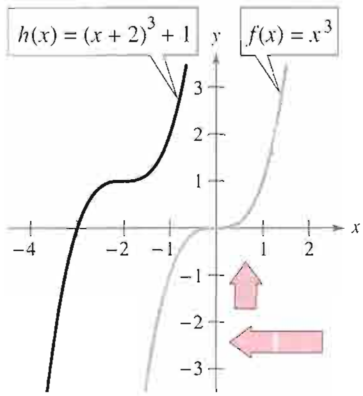


FIGURE 2.50

In Figure 2.50, notice that the same result is obtained if the vertical shift precedes the horizontal shift *or* if the horizontal shift precedes the vertical shift.

Exploration

Graphing utilities are ideal tools for exploring translations of functions. Graph f, g, and h in same viewing window. Before looking at the graphs, try to predict how the graphs of g and h relate to the graph of f.

a. $f(x) = x^2$, $g(x) = (x - 4)^2$, $h(x) = (x - 4)^2 + 3$

b. $f(x) = x^2$, $g(x) = (x + 1)^2$, $h(x) = (x + 1)^2 - 2$

c. $f(x) = x^2$, $g(x) = (x + 4)^2$, $h(x) = (x + 4)^2 + 2$

Reflecting Graphs

The second common type of transformation is a **reflection.** For instance, if you consider the x-axis to be a mirror, the graph of

$$h(x) = -x^2$$

is the mirror image (or reflection) of the graph of $f(x) = x^2$, as shown in Figure 2.51.

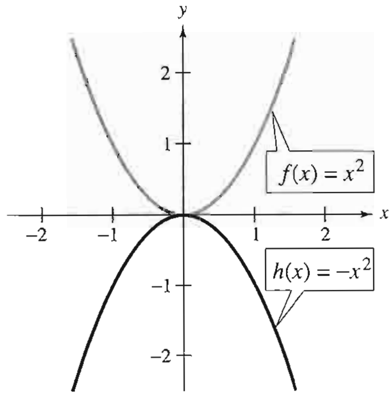


FIGURE 2.51

Reflections in the Coordinate Axes

Reflections in the coordinate axes of the graph of $y = f(x)$ are represented as follows.

1. Reflection in the x-axis: $h(x) = -f(x)$
2. Reflection in the y-axis: $h(x) = f(-x)$

Example 2 ▶ Finding Equations from Graphs

The graph of the function

$$f(x) = x^4$$

is shown in Figure 2.52. Each of the graphs in Figure 2.53 is a transformation of the graph of f. Find an equation for each of these functions.

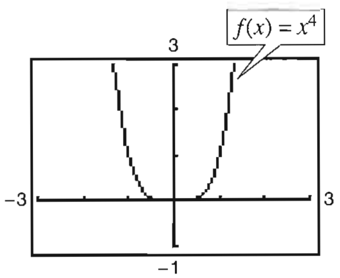
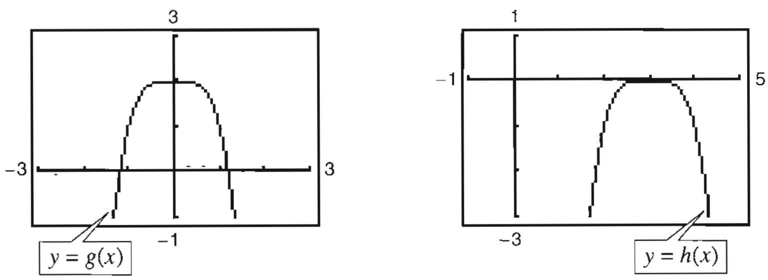


FIGURE 2.52



(a) (b)

FIGURE 2.53

Solution

a. The graph of g is a reflection in the x-axis *followed by* an upward shift of two units of the graph of $f(x) = x^4$. So, the equation for g is

$$g(x) = -x^4 + 2.$$

b. The graph of h is a horizontal shift of three units to the right *followed by* a reflection in the x-axis of the graph of $f(x) = x^4$. So, the equation for h is

$$h(x) = -(x - 3)^4.$$

Exploration

Reverse the order of transformations in Example 2(a). Do you obtain the same graph? Do the same for Example 2(b). Do you obtain the same graph? Explain.

Example 3 ▶ Reflections and Shifts

Compare the graph of each function with the graph of $f(x) = \sqrt{x}$.

a. $g(x) = -\sqrt{x}$ **b.** $h(x) = \sqrt{-x}$ **c.** $k(x) = -\sqrt{x+2}$

Solution

a. The graph of g is a reflection of the graph of f in the x-axis because

$$\begin{aligned} g(x) &= -\sqrt{x} \\ &= -f(x). \end{aligned}$$

The graph of g compared with f is shown in Figure 2.54.

b. The graph of h is a reflection of the graph of f in the y-axis because

$$\begin{aligned} h(x) &= \sqrt{-x} \\ &= f(-x). \end{aligned}$$

The graph of h compared with f is shown in Figure 2.55.

c. The graph of k is a left shift of two units, followed by a reflection in the x-axis because

$$\begin{aligned} k(x) &= -\sqrt{x+2} \\ &= -f(x+2). \end{aligned}$$

The graph of k compared with f is shown in Figure 2.56.

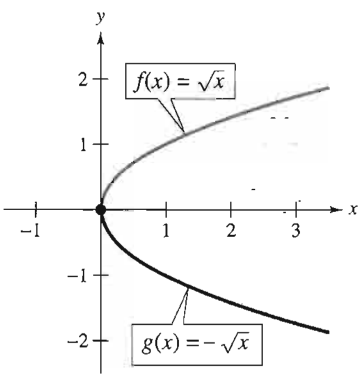


FIGURE 2.54

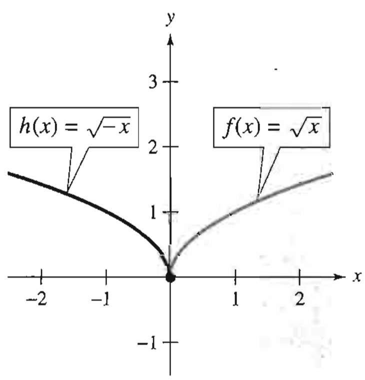


FIGURE 2.55

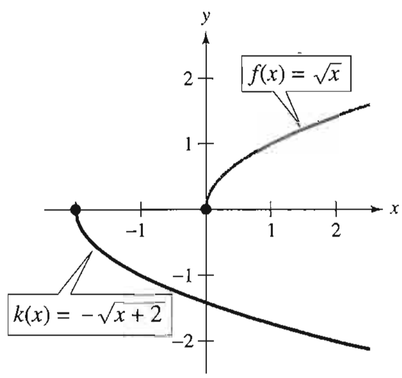


FIGURE 2.56

When sketching the graphs of functions involving square roots, remember that the domain must be restricted to exclude negative numbers inside the radical. For instance, here are the domains of the functions in Example 3.

Domain of $g(x) = -\sqrt{x}$: $x \geq 0$

Domain of $h(x) = \sqrt{-x}$: $x \leq 0$

Domain of $k(x) = -\sqrt{x+2}$: $x \geq -2$

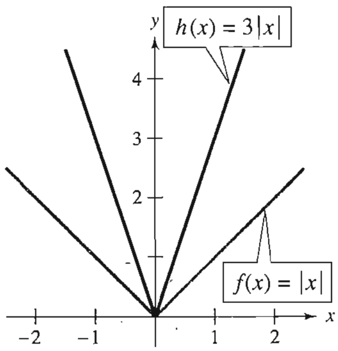


FIGURE 2.57

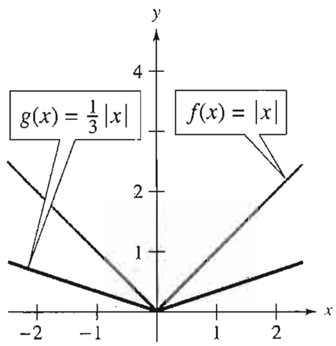


FIGURE 2.58

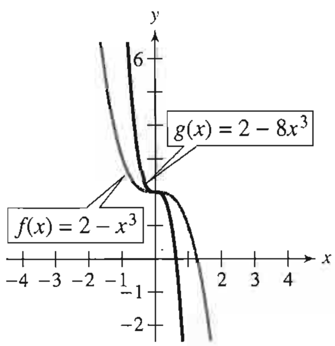


FIGURE 2.59

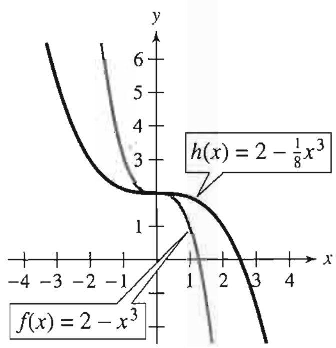


FIGURE 2.60

Nonrigid Transformations

Horizontal shifts, vertical shifts, and reflections are **rigid transformations** because the basic shape of the graph is unchanged. These transformations change only the *position* of the graph in the xy-plane. **Nonrigid transformations** are those that cause a *distortion*—a change in the shape of the original graph. For instance, a nonrigid transformation of the graph of $y = f(x)$ is represented by $g(x) = cf(x)$, where the transformation is a **vertical stretch** if $c > 1$ and a **vertical shrink** if $0 < c < 1$. Another nonrigid transformation of the graph of $y = f(x)$ is represented by $h(x) = f(cx)$, where the transformation is a **horizontal shrink** if $c > 1$ and a **horizontal stretch** if $0 < c < 1$.

Example 4 ▶ Nonrigid Transformations

Compare the graph of each function with the graph of $f(x) = |x|$.

a. $h(x) = 3|x|$ **b.** $g(x) = \frac{1}{3}|x|$

Solution

a. Relative to the graph of $f(x) = |x|$, the graph of

$$h(x) = 3|x| = 3f(x)$$

is a vertical stretch (each y-value is multiplied by 3) of the graph of f. (See Figure 2.57.)

b. Similarly, the graph of

$$g(x) = \frac{1}{3}|x| = \frac{1}{3}f(x)$$

is a vertical shrink $\left(\text{each } y\text{-value is multiplied by } \frac{1}{3}\right)$ of the graph of f. (See Figure 2.58.)

Example 5 ▶ Nonrigid Transformations

Compare the graph of each function with the graph of $f(x) = 2 - x^3$.

a. $g(x) = f(2x)$ **b.** $h(x) = f\left(\frac{1}{2}x\right)$

Solution

a. Relative to the graph of $f(x) = 2 - x^3$, the graph of

$$g(x) = f(2x) = 2 - (2x)^3 = 2 - 8x^3$$

is a horizontal shrink $\left(\text{each } x\text{-value is multiplied by } \frac{1}{2}\right)$ of the graph of f. (See Figure 2.59.)

b. Similarly, the graph of

$$h(x) = f\left(\tfrac{1}{2}x\right) = 2 - \left(\tfrac{1}{2}x\right)^3 = 2 - \tfrac{1}{8}x^3$$

is a horizontal stretch (each x-value is multiplied by 2) of the graph of f. (See Figure 2.60.)

2.5 Exercises

1. For each function, sketch (on the same set of coordinate axes) a graph of each function for $c = -1, 1,$ and 3.

(a) $f(x) = |x| + c$ (b) $f(x) = |x - c|$

(c) $f(x) = |x + 4| + c$

2. For each function, sketch (on the same set of coordinate axes) a graph of each function for $c = -3, -1, 1,$ and 3.

(a) $f(x) = \sqrt{x} + c$ (b) $f(x) = \sqrt{x - c}$

(c) $f(x) = \sqrt{x - 3} + c$

3. For each function, sketch (on the same set of coordinate axes) a graph of each function for $c = -2, 0,$ and 2.

(a) $f(x) = [\![x]\!] + c$ (b) $f(x) = [\![x + c]\!]$

(c) $f(x) = [\![x - 1]\!] + c$

4. For each function, sketch (on the same set of coordinate axes) a graph of each function for $c = -3, -1, 1,$ and 3.

(a) $f(x) = \begin{cases} x^2 + c, & x < 0 \\ -x^2 + c, & x \geq 0 \end{cases}$

(b) $f(x) = \begin{cases} (x + c)^2, & x < 0 \\ -(x + c)^2, & x \geq 0 \end{cases}$

5. Use the graph of f to sketch each graph. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*

(a) $y = f(x) + 2$

(b) $y = f(x - 2)$

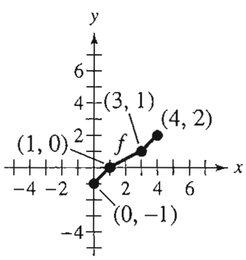
(c) $y = 2f(x)$

(d) $y = -f(x)$

(e) $y = f(x + 3)$

(f) $y = f(-x)$

(g) $y = f\left(\frac{1}{2}x\right)$



6. Use the graph of f to sketch each graph. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*

(a) $y = f(-x)$

(b) $y = f(x) + 4$

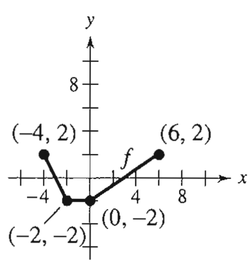
(c) $y = 2f(x)$

(d) $y = -f(x - 4)$

(e) $y = f(x) - 3$

(f) $y = -f(x) - 1$

(g) $y = f(2x)$



7. Use the graph of f to sketch each graph. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*

(a) $y = f(x) - 1$

(b) $y = f(x - 1)$

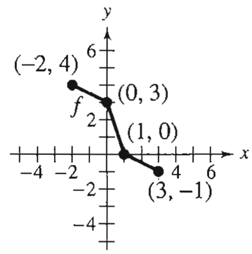
(c) $y = f(-x)$

(d) $y = f(x + 1)$

(e) $y = -f(x - 2)$

(f) $y = \frac{1}{2}f(x)$

(g) $y = f(2x)$



8. Use the graph of f to sketch each graph. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*

(a) $y = f(x - 5)$

(b) $y = -f(x) + 3$

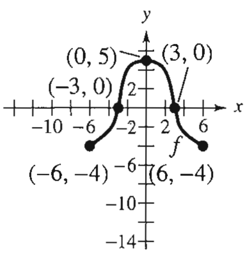
(c) $y = \frac{1}{3}f(x)$

(d) $y = -f(x + 1)$

(e) $y = f(-x)$

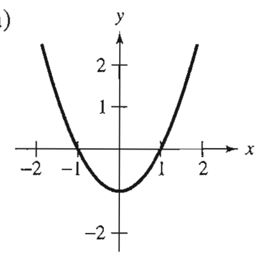
(f) $y = f(x) - 10$

(g) $y = f\left(\frac{1}{3}x\right)$

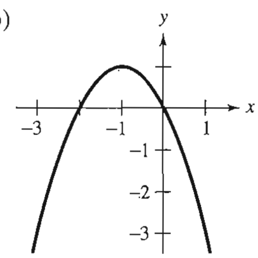


9. Use the graph of $f(x) = x^2$ to write an equation for each function whose graph is shown.

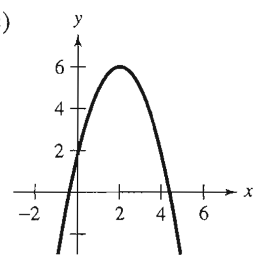
(a)



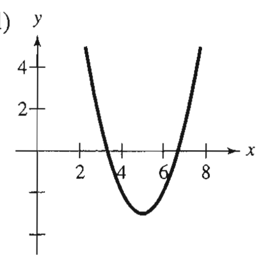
(b)



(c)

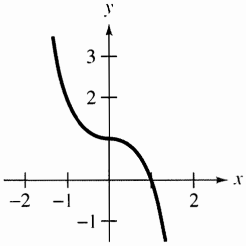


(d)

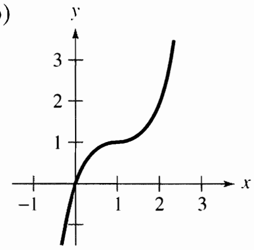


10. Use the graph of $f(x) = x^3$ to write an equation for each function whose graph is shown.

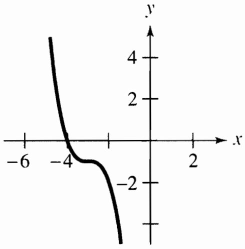
(a)



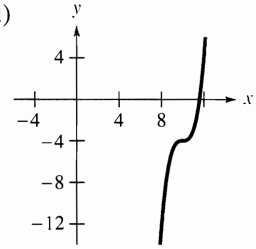
(b)



(c)

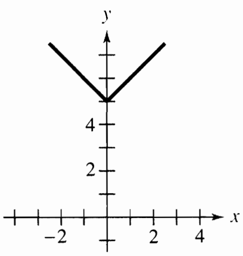


(d)

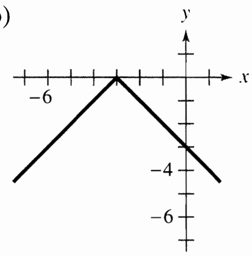


11. Use the graph of $f(x) = |x|$ to write an equation for each function whose graph is shown.

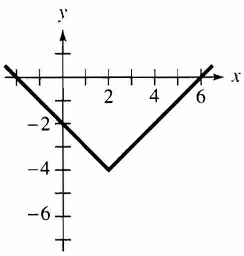
(a)



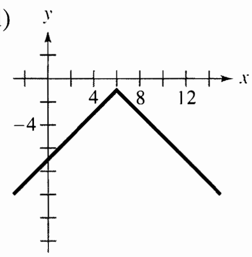
(b)



(c)

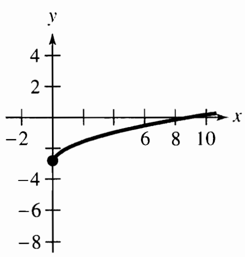


(d)

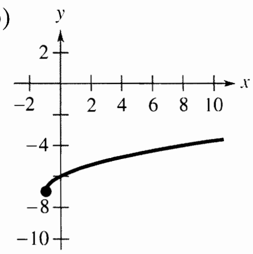


12. Use the graph of $f(x) = \sqrt{x}$ to write an equation for each function whose graph is shown.

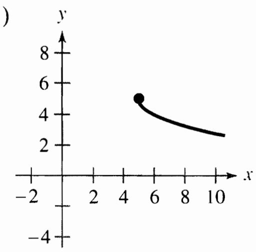
(a)



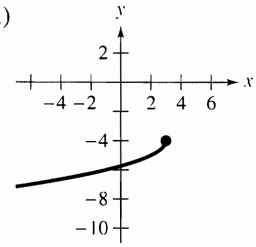
(b)



(c)

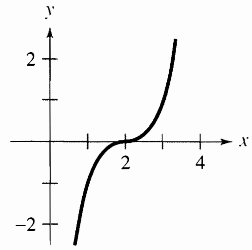


(d)

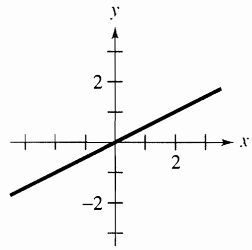


In Exercises 13–18, identify the common function and the transformation shown in the graph. Write an equation for the function shown in the graph.

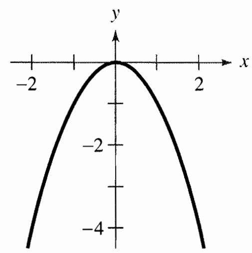
13.



14.



15.



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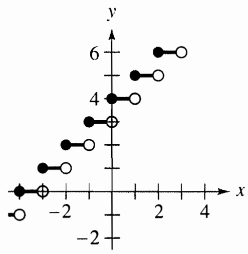
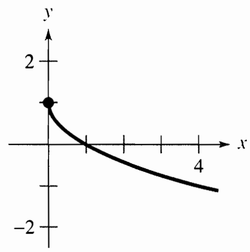
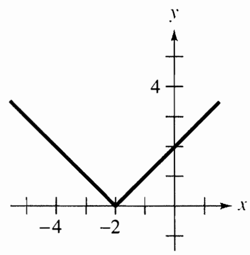


17.



18.



In Exercises 19–38, describe the transformation from a common function that occurs in the function. Then sketch its graph.

19. $f(x) = 12 - x^2$

20. $f(x) = (x - 8)^2$

21. $f(x) = x^3 + 7$

22. $f(x) = -x^3 - 1$

23. $f(x) = 2 - (x + 5)^2$

24. $f(x) = -(x + 10)^2 + 5$

25. $f(x) = (x - 1)^3 + 2$

26. $f(x) = (x + 3)^3 - 10$

27. $f(x) = -|x| - 2$

28. $f(x) = 6 - |x + 5|$

29. $f(x) = -|x + 4| + 8$

30. $f(x) = |-x + 3| + 9$

31. $f(x) = 3 - [\![x]\!]$

32. $f(x) = 2[\![x + 5]\!]$

33. $f(x) = \sqrt{x - 9}$ **34.** $f(x) = \sqrt{x + 4} + 8$

35. $f(x) = \sqrt{7 - x} - 2$ **36.** $f(x) = -\sqrt{x + 1} - 6$

37. $f(x) = \sqrt{\frac{1}{2}x} - 4$ **38.** $f(x) = \sqrt{3x} + 1$

In Exercises 39–46, write an equation for the function that is described by the given characteristics.

39. The shape of $f(x) = x^2$, but moved two units to the right and eight units downward

40. The shape of $f(x) = x^2$, but moved three units to the left, seven units upward, and reflected in the x-axis

41. The shape of $f(x) = x^3$, but moved 13 units to the right

42. The shape of $f(x) = x^3$, but moved six units to the left, six units downward, and reflected in the y-axis

43. The shape of $f(x) = |x|$, but moved 10 units upward and reflected in the x-axis

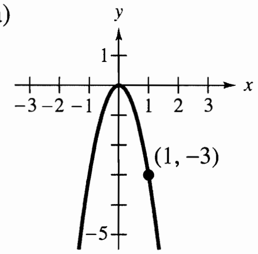
44. The shape of $f(x) = |x|$, but moved one unit to the left and seven units downward

45. The shape of $f(x) = \sqrt{x}$, but moved six units to the left and reflected in both the x-axis and the y-axis

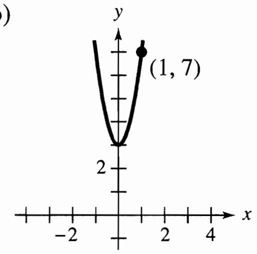
46. The shape of $f(x) = \sqrt{x}$, but moved nine units downward and reflected in both the x-axis and the y-axis

47. Use the graph of $f(x) = x^2$ to write an equation for each function whose graph is shown.

(a)

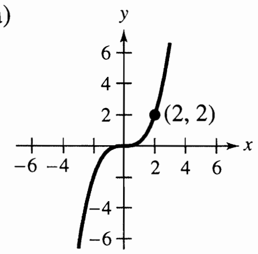


(b)

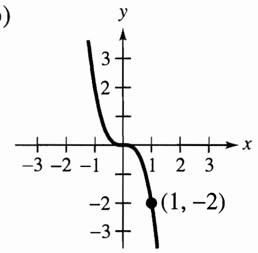


48. Use the graph of $f(x) = x^3$ to write an equation for each function whose graph is shown.

(a)

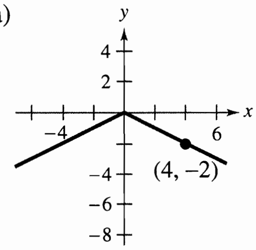


(b)

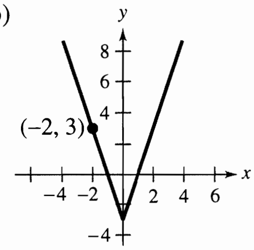


49. Use the graph of $f(x) = |x|$ to write an equation for each function whose graph is shown.

(a)

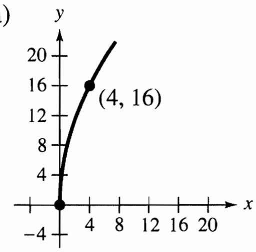


(b)

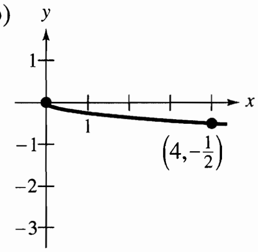


50. Use the graph of $f(x) = \sqrt{x}$ to write an equation for each function whose graph is shown.

(a)

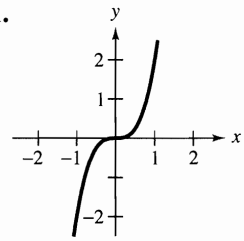


(b)

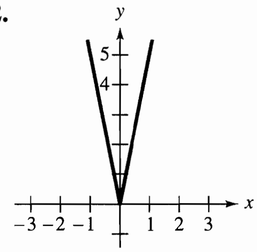


In Exercises 51–56, identify the common function and the transformation shown in the graph. Write an equation for the function shown in the graph. Then use a graphing utility to verify your answer.

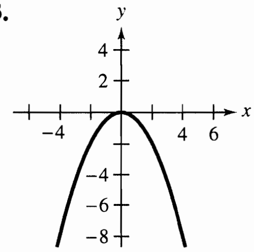
51.



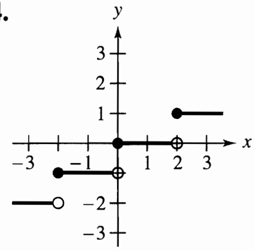
52.



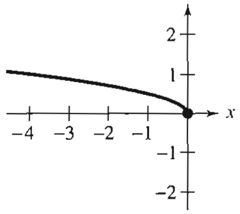
53.



54.



55.



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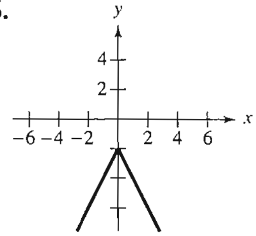
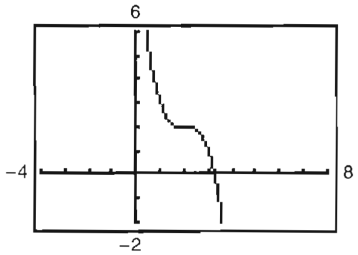


Graphical Analysis In Exercises 57–60, use the viewing window shown to write a possible equation for the transformation of the common function.

57.



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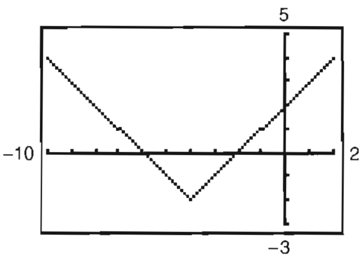

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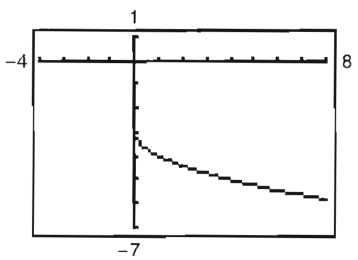

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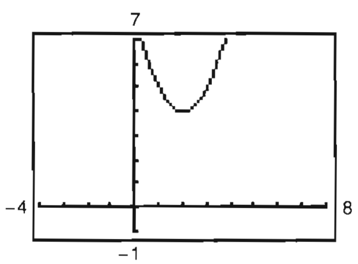
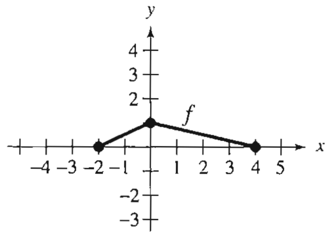


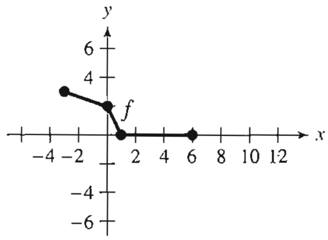
Graphical Reasoning In Exercises 61 and 62, use the graph of f to sketch the graph of g. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

61.



(a) $g(x) = f(x) + 2$ (b) $g(x) = f(x) - 1$
(c) $g(x) = f(-x)$ (d) $g(x) = -2f(x)$
(e) $g(x) = f(4x)$ (f) $g(x) = f\left(\frac{1}{2}x\right)$

62.



(a) $g(x) = f(x) - 5$ (b) $g(x) = f(x) + \frac{1}{2}$
(c) $g(x) = f(-x)$ (d) $g(x) = -4f(x)$
(e) $g(x) = f(2x) + 1$ (f) $g(x) = f\left(\frac{1}{4}x\right) - 2$

▶ Model It

63. ***Fuel Use*** The amount of fuel F (in billions of gallons) used by trucks from 1980 through 1999 can be approximated by the function

$$F = f(t) = 20.5 + 0.035t^2$$

where $t = 0$ represents 1980. (Source: U.S. Federal Highway Administration)

(a) Describe the transformation of the common function $f(x) = x^2$. Then sketch the graph over the interval $0 \le t \le 19$.

(b) Find and interpret $\dfrac{f(19) - f(0)}{19 - 0}$.

(c) Rewrite the function so that $t = 0$ represents 1990. Explain how you got your answer.

(d) Use the model from part (c) to predict the amount of fuel used by trucks in 2005. Does your answer seem reasonable? Explain.

64. ***Finance*** The amount M (in trillions of dollars) of mortgage debt outstanding in the United States from 1980 through 1999 can be approximated by the function $M = f(t) = 0.0037(t + 14.979)^2$, where $t = 0$ represents 1980. (Source: Board of Governors of the Federal Reserve System)

(a) Describe the transformation of the common function $f(x) = x^2$. Then sketch the graph over the interval $0 \le t \le 19$.

(b) Rewrite the function so that $t = 0$ represents 1990. Explain how you got your answer.

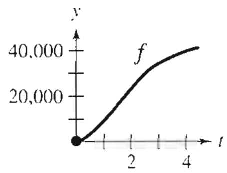
Synthesis

True or False? **In Exercises 65 and 66, determine whether the statement is true or false. Justify your answer.**

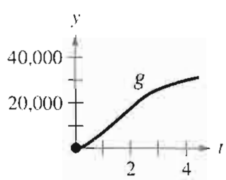
65. The graphs of $f(x) = |x| + 6$ and $f(x) = |-x| + 6$ are identical.

66. If the graph of the common function $f(x) = x^2$ is moved six units to the right, three units upward, and reflected in the x-axis, then the point $(-2, 19)$ will lie on the graph of the transformation.

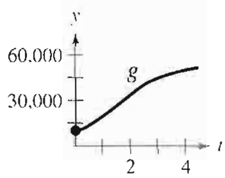
67. ***Describing Profits*** Management originally predicted that the profits from the sales of a new product would be approximated by the graph of the function f shown. The actual profits are shown by the function g along with a verbal description. Use the concepts of transformations of graphs to write g in terms of f.



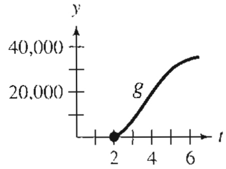
(a) The profits were only three-fourths as large as expected.



(b) The profits were consistently \$10,000 greater than predicted.



(c) There was a two-year delay in the introduction of the product. After sales began, profits grew as expected.



68. Explain why the graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ about the x-axis.

69. The graph of $y = f(x)$ passes through the points $(0, 1)$, $(1, 2)$, and $(2, 3)$. Find the corresponding points on the graph of $y = f(x + 2) - 1$.

70. ***Think About It*** You can use either of two methods to graph a function: plotting points or translating a common function as shown in this section. Which method of graphing do you prefer to use for each function? Explain.

(a) $f(x) = 3x^2 - 4x + 1$

(b) $f(x) = 2(x - 1)^2 - 6$

Review

In Exercises 71–78, perform the operation and simplify.

71. $\dfrac{4}{x} + \dfrac{4}{1 - x}$

72. $\dfrac{2}{x + 5} - \dfrac{2}{x - 5}$

73. $\dfrac{3}{x - 1} - \dfrac{2}{x(x - 1)}$

74. $\dfrac{x}{x - 5} + \dfrac{1}{2}$

75. $(x - 4)\left(\dfrac{1}{\sqrt{x^2 - 4}}\right)$

76. $\left(\dfrac{x}{x^2 - 4}\right)\left(\dfrac{x^2 - x - 2}{x^2}\right)$

77. $(x^2 - 9) \div \left(\dfrac{x + 3}{5}\right)$

78. $\left(\dfrac{x}{x^2 - 3x - 28}\right) \div \left(\dfrac{x^2 + 3x}{x^2 + 5x + 4}\right)$

In Exercises 79 and 80, evaluate the function at the specified values of the independent variable and simplify.

79. $f(x) = x^2 - 6x + 11$

(a) $f(-3)$ (b) $f\left(-\frac{1}{2}\right)$ (c) $f(x - 3)$

80. $f(x) = \sqrt{x + 10} - 3$

(a) $f(-10)$ (b) $f(26)$ (c) $f(x - 10)$

In Exercises 81–84, find the domain of the function.

81. $f(x) = \dfrac{2}{11 - x}$

82. $f(x) = \dfrac{\sqrt{x - 3}}{x - 8}$

83. $f(x) = \sqrt{81 - x^2}$

84. $f(x) = \sqrt[3]{4 - x^2}$

2.6 Combinations of Functions

▶ **What you should learn**

- How to add, subtract, multiply, and divide functions
- How to find the composition of one function with another function
- How to use combinations of functions to model and solve real-life problems

▶ **Why you should learn it**

Combinations of functions can be used to model and solve real-life problems. For instance, in Exercise 33 on page 235, combinations of functions are used to analyze U.S. health expenditures.

Charles Gupton/Tony Stone Images

Arithmetic Combinations of Functions

Just as two real numbers can be combined by the operations of addition, subtraction, multiplication, and division to form other real numbers, two *functions* can be combined to create new functions. For example, the functions $f(x) = 2x - 3$ and $g(x) = x^2 - 1$ can be combined to form the sum, difference, product, and quotient of f and g.

$$f(x) + g(x) = (2x - 3) + (x^2 - 1)$$
$$= x^2 + 2x - 4 \qquad \text{Sum}$$
$$f(x) - g(x) = (2x - 3) - (x^2 - 1)$$
$$= -x^2 + 2x - 2 \qquad \text{Difference}$$
$$f(x)g(x) = (2x - 3)(x^2 - 1)$$
$$= 2x^3 - 3x^2 - 2x + 3 \qquad \text{Product}$$
$$\frac{f(x)}{g(x)} = \frac{2x - 3}{x^2 - 1}, \quad x \neq \pm 1 \qquad \text{Quotient}$$

The domain of an **arithmetic combination** of functions f and g consists of all real numbers that are common to the domains of f and g. In the case of the quotient $f(x)/g(x)$, there is the further restriction that $g(x) \neq 0$.

Sum, Difference, Product, and Quotient of Functions

Let f and g be two functions with overlapping domains. Then, for all x common to both domains, the *sum*, *difference*, *product*, and *quotient* of f and g are defined as follows.

1. *Sum:* $(f + g)(x) = f(x) + g(x)$
2. *Difference:* $(f - g)(x) = f(x) - g(x)$
3. *Product:* $(fg)(x) = f(x) \cdot g(x)$
4. *Quotient:* $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$

Example 1 ▶ **Finding the Sum of Two Functions**

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f + g)(x)$.

Solution

$$(f + g)(x) = f(x) + g(x) = (2x + 1) + (x^2 + 2x - 1) = x^2 + 4x$$

Example 2 ▶ Finding the Difference of Two Functions

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f - g)(x)$. Then evaluate the difference when $x = 2$.

Solution

The difference of f and g is

$$\begin{aligned}(f - g)(x) &= f(x) - g(x)\\ &= (2x + 1) - (x^2 + 2x - 1)\\ &= -x^2 + 2.\end{aligned}$$

When $x = 2$, the value of this difference is

$$\begin{aligned}(f - g)(2) &= -(2)^2 + 2\\ &= -2.\end{aligned}$$

In Examples 1 and 2, both f and g have domains that consist of all real numbers. So, the domains of $(f + g)$ and $(f - g)$ are also the set of all real numbers. Remember that any restrictions on the domains of f and g must be considered when forming the sum, difference, product, or quotient of f and g.

Example 3 ▶ Finding the Domains of Quotients of Functions

Find the domains of $\left(\frac{f}{g}\right)(x)$ and $\left(\frac{g}{f}\right)(x)$ for the functions

$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = \sqrt{4 - x^2}.$$

Solution

The quotient of f and g is

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{4 - x^2}}$$

and the quotient of g and f is

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{4 - x^2}}{\sqrt{x}}.$$

The domain of f is $[0, \infty)$ and the domain of g is $[-2, 2]$. The intersection of these domains is $[0, 2]$. So, the domains of $\left(\frac{f}{g}\right)$ and $\left(\frac{g}{f}\right)$ are as follows.

$$\text{Domain of } \left(\frac{f}{g}\right): [0, 2) \qquad \text{Domain of } \left(\frac{g}{f}\right): (0, 2]$$

Can you see why these two domains differ slightly?

Composition of Functions

Another way of combining two functions is to form the **composition** of one with the other. For instance, if $f(x) = x^2$ and $g(x) = x + 1$, the composition of f with g is

$$f(g(x)) = f(x + 1)$$
$$= (x + 1)^2.$$

This composition is denoted as $(f \circ g)$.

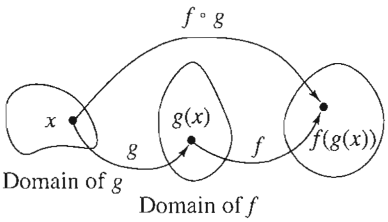


FIGURE 2.61

> **Definition of Composition of Two Functions**
>
> The **composition** of the function f with the function g is
>
> $$(f \circ g)(x) = f(g(x)).$$
>
> The domain of $(f \circ g)$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. (See Figure 2.61.)

Example 4 ▶ Composition of Functions

Given $f(x) = x + 2$ and $g(x) = 4 - x^2$, find the following.

a. $(f \circ g)(x)$ **b.** $(g \circ f)(x)$ **c.** $(g \circ f)(-2)$

Solution

a. The composition of f with g is as follows.

$(f \circ g)(x) = f(g(x))$	Definition of $f \circ g$
$= f(4 - x^2)$	Definition of $g(x)$
$= (4 - x^2) + 2$	Definition of $f(x)$
$= -x^2 + 6$	Simplify.

b. The composition of g with f is as follows.

$(g \circ f)(x) = g(f(x))$	Definition of $g \circ f$
$= g(x + 2)$	Definition of $f(x)$
$= 4 - (x + 2)^2$	Definition of $g(x)$
$= 4 - (x^2 + 4x + 4)$	Expand.
$= -x^2 - 4x$	Simplify.

Note that, in this case, $(f \circ g)(x) \neq (g \circ f)(x)$.

c. Using the result of part (b), you can write the following.

$(g \circ f)(-2) = -(-2)^2 - 4(-2)$	Substitute.
$= -4 + 8$	Simplify.
$= 4$	Simplify.

STUDY TIP

The following tables of values help illustrate the composition $(f \circ g)(x)$ given in Example 4.

x	0	1	2	3
$g(x)$	4	3	0	-5

$g(x)$	4	3	0	-5
$f(g(x))$	6	5	2	-3

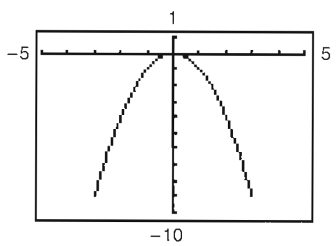
x	0	1	2	3
$f(g(x))$	6	5	2	-3

Note that the first two tables can be combined (or "composed") to produce the values given in the third table.

Technology You can use a graphing utility to determine the domain of a composition of functions. For the composition in Example 5, enter the function composition as

$$y = \left(\sqrt{9 - x^2}\right)^2 - 9.$$

You should obtain the graph shown below. Use the *trace* feature to determine that the x-coordinates of points on the graph extend from -3 to 3. So, the domain of $(f \circ g)(x)$ is $-3 \le x \le 3$.



Example 5 ▶ Finding the Domain of a Composite Function

Find the composition $(f \circ g)(x)$ for the functions

$$f(x) = x^2 - 9 \quad \text{and} \quad g(x) = \sqrt{9 - x^2}.$$

Then find the domain of $(f \circ g)$.

Solution

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\sqrt{9 - x^2}\right) \\ &= \left(\sqrt{9 - x^2}\right)^2 - 9 \\ &= 9 - x^2 - 9 \\ &= -x^2\end{aligned}$$

From this, it might appear that the domain of the composition is the set of all real numbers. Because the domain of f is the set of all real numbers and the domain of g is $-3 \le x \le 3$, the domain of $(f \circ g)$ is $-3 \le x \le 3$.

In Examples 4 and 5, you formed the composition of two given functions. In calculus, it is also important to be able to identify two functions that make up a given composite function. For instance, the function h given by

$$h(x) = (3x - 5)^3$$

is the composition of f with g, where $f(x) = x^3$ and $g(x) = 3x - 5$. That is,

$$h(x) = (3x - 5)^3 = [g(x)]^3 = f(g(x)).$$

Basically, to "decompose" a composite function, look for an "inner" function and an "outer" function. In the function h above, $g(x) = 3x - 5$ is the inner function and $f(x) = x^3$ is the outer function.

Example 6 ▶ Finding Components of Composite Functions

Express the function $h(x) = \dfrac{1}{(x - 2)^2}$ as a composition of two functions.

Solution

One way to write h as a composition of two functions is to take the inner function to be $g(x) = x - 2$ and the outer function to be

$$f(x) = \frac{1}{x^2} = x^{-2}.$$

Then you can write

$$h(x) = \frac{1}{(x - 2)^2} = (x - 2)^{-2} = f(x - 2) = f(g(x)).$$

Exploration

You are buying an automobile whose price is \$18,500. Which of the following options would you choose? Explain.

a. You are given a factory rebate of \$2000, followed by a dealer discount of 10%.

b. You are given a dealer discount of 10%, followed by a factory rebate of \$2000.

Let $f(x) = x - 2000$ and let $g(x) = 0.9x$. Which option is represented by the composite $f(g(x))$? Which is represented by the composite $g(f(x))$?

Application

Example 7 ▶ Bacteria Count

The number N of bacteria in a refrigerated food is

$$N(T) = 20T^2 - 80T + 500, \qquad 2 \le T \le 14$$

where T is the temperature of the food in degrees Celsius. When the food is removed from refrigeration, the temperature is

$$T(t) = 4t + 2, \qquad 0 \le t \le 3$$

where t is the time in hours. (a) Find the composite $N(T(t))$ and interpret its meaning in context. (b) Find the time when the bacterial count reaches 2000.

Solution

a.
$$\begin{aligned} N(T(t)) &= 20(4t + 2)^2 - 80(4t + 2) + 500 \\ &= 20(16t^2 + 16t + 4) - 320t - 160 + 500 \\ &= 320t^2 + 320t + 80 - 320t - 160 + 500 \\ &= 320t^2 + 420 \end{aligned}$$

The composite function $N(T(t))$ represents the number of bacteria in the food as a function of time.

b. The bacterial count will reach 2000 when $320t^2 + 420 = 2000$. Solve this equation to find that the count will reach 2000 when $t \approx 2.2$ hours. When you solve this equation, note that the negative value is rejected because it is not in the domain of the composite function.

Writing ABOUT MATHEMATICS

Analyzing Arithmetic Combinations of Functions

a. Use the graphs of f and $(f + g)$ in Figure 2.62 to make a table showing the values of $g(x)$ when $x = 1, 2, 3, 4, 5,$ and 6. Explain your reasoning.

b. Use the graphs of f and $(f - h)$ in Figure 2.62 to make a table showing the values of $h(x)$ when $x = 1, 2, 3, 4, 5,$ and 6. Explain your reasoning.

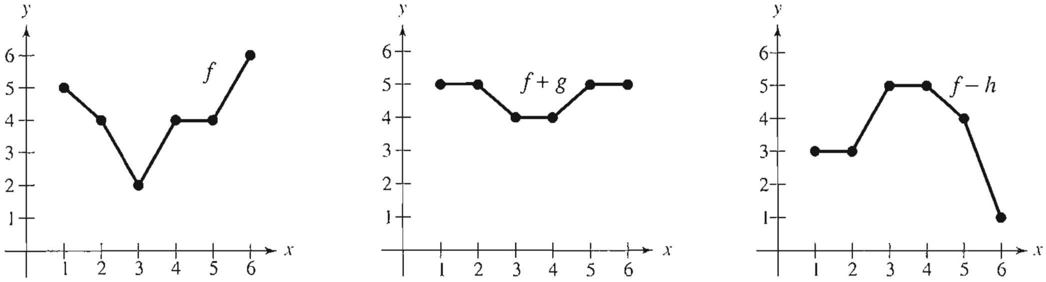
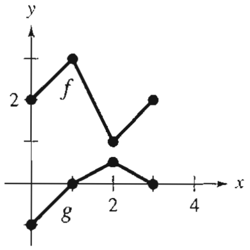


FIGURE 2.62

2.6 Exercises

In Exercises 1–4, use the graphs of f and g to graph $h(x) = (f + g)(x)$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

1.



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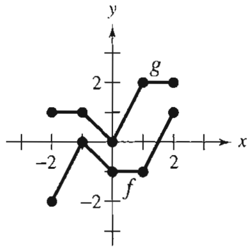

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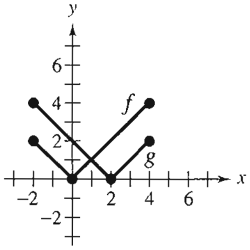

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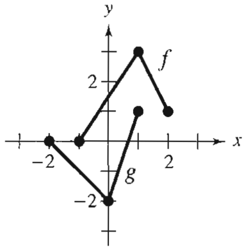


In Exercises 5–12, find (a) $(f + g)(x)$, (b) $(f - g)(x)$, (c) $(fg)(x)$, and (d) $(f/g)(x)$. What is the domain of f/g?

5. $f(x) = x + 2, \quad g(x) = x - 2$

6. $f(x) = 2x - 5, \quad g(x) = 2 - x$

7. $f(x) = x^2, \quad g(x) = 4x - 5$

8. $f(x) = 2x - 5, \quad g(x) = 4$

9. $f(x) = x^2 + 6, \quad g(x) = \sqrt{1 - x}$

10. $f(x) = \sqrt{x^2 - 4}, \quad g(x) = \dfrac{x^2}{x^2 + 1}$

11. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x^2}$

12. $f(x) = \dfrac{x}{x + 1}, \quad g(x) = x^3$

In Exercises 13–24, evaluate the indicated function for $f(x) = x^2 + 1$ and $g(x) = x - 4$.

13. $(f + g)(2)$ **14.** $(f - g)(-1)$

15. $(f - g)(0)$ **16.** $(f + g)(1)$

17. $(f - g)(3t)$ **18.** $(f + g)(t - 2)$

19. $(fg)(6)$ **20.** $(fg)(-6)$

21. $\left(\dfrac{f}{g}\right)(5)$ **22.** $\left(\dfrac{f}{g}\right)(0)$

23. $\left(\dfrac{f}{g}\right)(-1) - g(3)$ **24.** $(fg)(5) + f(4)$

In Exercises 25–28, graph the functions f, g, and $f + g$ on the same set of coordinate axes.

25. $f(x) = \frac{1}{2}x, \quad g(x) = x - 1$

26. $f(x) = \frac{1}{3}x, \quad g(x) = -x + 4$

27. $f(x) = x^2, \quad g(x) = -2x$

28. $f(x) = 4 - x^2, \quad g(x) = x$

Graphical Reasoning In Exercises 29 and 30, use a graphing utility to graph f, g, and $f + g$ in the same viewing window. Which function contributes most to the magnitude of the sum when $0 \le x \le 2$? Which function contributes most to the magnitude of the sum when $x > 6$?

29. $f(x) = 3x, \quad g(x) = -\dfrac{x^3}{10}$

30. $f(x) = \dfrac{x}{2}, \quad g(x) = \sqrt{x}$

31. ***Stopping Distance*** The research and development department of an automobile manufacturer has determined that when required to stop quickly to avoid an accident, the distance (in feet) a car travels during the driver's reaction time is given by $R(x) = \frac{3}{4}x$, where x is the speed of the car in miles per hour. The distance (in feet) traveled while the driver is braking is $B(x) = \frac{1}{15}x^2$. Find the function that represents the total stopping distance T. Graph the functions R, B, and T on the same set of coordinate axes for $0 \le x \le 60$.

32. ***Sales*** From 1997 to 2002, the sales R_1 (in thousands of dollars) for one of two restaurants owned by the same parent company can be modeled by

$$R_1 = 480 - 8t - 0.8t^2, \qquad t = 0, 1, 2, 3, 4, 5$$

where $t = 0$ represents 1997. During the same six-year period, the sales R_2 (in thousands of dollars) for the second restaurant can be modeled by

$$R_2 = 254 + 0.78t, \qquad t = 0, 1, 2, 3, 4, 5.$$

Write a function that represents the total sales of the two restaurants owned by the same parent company. Use a graphing utility to graph the total sales function.

▶ Model It

33. ***Health Care Costs*** The table shows the total amount (in billions of dollars) spent on health services and supplies in the United States (including Puerto Rico) for the years 1993 through 1999. The variables y_1, y_2, and y_3 represent out-of-pocket payments, insurance premiums, and other types of payments, respectively. (Source: Centers for Medicare and Medicaid Services)

Year	y_1	y_2	y_3
1993	148.9	295.7	39.1
1994	146.2	308.9	40.8
1995	149.2	322.3	44.8
1996	155.0	337.4	47.9
1997	165.5	355.6	52.0
1998	176.1	376.8	54.8
1999	186.5	401.2	58.9

(a) Use the *regression* feature of a graphing utility to find a quadratic model for y_1 and linear models for y_2 and y_3. Let $t = 3$ represent 1993.

(b) Find $y_1 + y_2 + y_3$. What does this sum represent?

(c) Use a graphing utility to graph y_1, y_2, y_3, and $y_1 + y_2 + y_3$ in the same viewing window.

(d) Use the model from part (b) to estimate the total amount spent on health services and supplies in the years 2003 and 2005.

34. ***Graphical Reasoning*** An electronically controlled thermostat in a home is programmed to lower the temperature automatically during the night. The temperature in the house T (in degrees Fahrenheit) is given in terms of t, the time in hours on a 24-hour clock (see figure).

(a) Explain why T is a function of t.

(b) Approximate $T(4)$ and $T(15)$.

(c) The thermostat is reprogrammed to produce a temperature H for which $H(t) = T(t - 1)$. How does this change the temperature?

(d) The thermostat is reprogrammed to produce a temperature H for which $H(t) = T(t) - 1$. How does this change the temperature?

(e) Write a piecewise-defined function that represents the graph.

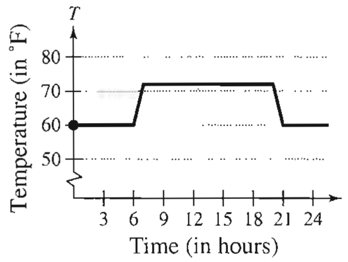


FIGURE FOR 34.

In Exercises 35–38, find (a) $f \circ g$, (b) $g \circ f$, and (c) $f \circ f$.

35. $f(x) = x^2$, $\quad g(x) = x - 1$

36. $f(x) = 3x + 5$, $\quad g(x) = 5 - x$

37. $f(x) = \sqrt[3]{x - 1}$, $\quad g(x) = x^3 + 1$

38. $f(x) = x^3$, $\quad g(x) = \dfrac{1}{x}$

In Exercises 39–46, find (a) $f \circ g$ and (b) $g \circ f$. Find the domain of each function and each composite function.

39. $f(x) = \sqrt{x + 4}$, $\quad g(x) = x^2$

40. $f(x) = \sqrt[3]{x - 5}$, $\quad g(x) = x^3 + 1$

41. $f(x) = x^2 + 1$, $\quad g(x) = \sqrt{x}$

42. $f(x) = x^{2/3}$, $\quad g(x) = x^6$

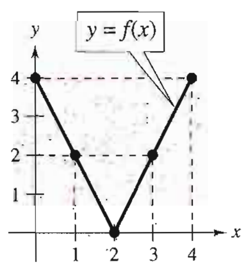
43. $f(x) = |x|$, $\quad g(x) = x + 6$

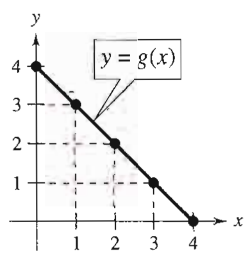
44. $f(x) = |x - 4|$, $\quad g(x) = 3 - x$

45. $f(x) = \dfrac{1}{x}$, $\quad g(x) = x + 3$

46. $f(x) = \dfrac{3}{x^2 - 1}$, $\quad g(x) = x + 1$

In Exercises 47–50, use the graphs of f and g to evaluate the functions.





47. (a) $(f + g)(3)$ $\quad$ (b) $(f/g)(2)$

48. (a) $(f - g)(1)$ $\quad$ (b) $(fg)(4)$

49. (a) $(f \circ g)(2)$ (b) $(g \circ f)(2)$

50. (a) $(f \circ g)(1)$ (b) $(g \circ f)(3)$

In Exercises 51–58, find two functions f and g such that $(f \circ g)(x) = h(x)$. (There is more than one correct answer.)

51. $h(x) = (2x + 1)^2$

52. $h(x) = (1 - x)^3$

53. $h(x) = \sqrt[3]{x^2 - 4}$

54. $h(x) = \sqrt{9 - x}$

55. $h(x) = \dfrac{1}{x + 2}$

56. $h(x) = \dfrac{4}{(5x + 2)^2}$

57. $h(x) = \dfrac{-x^2 + 3}{4 - x^2}$

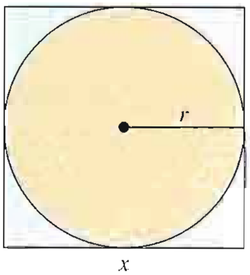
58. $h(x) = \dfrac{27x^3 + 6x}{10 - 27x^3}$

59. ***Geometry*** A square concrete foundation is prepared as a base for a cylindrical tank (see figure).

(a) Write the radius r of the tank as a function of the length x of the sides of the square.

(b) Write the area A of the circular base of the tank as a function of the radius r.

(c) Find and interpret $(A \circ r)(x)$.



60. ***Physics*** A pebble is dropped into a calm pond, causing ripples in the form of concentric circles (see figure). The radius (in feet) of the outer ripple is $r(t) = 0.6t$, where t is the time in seconds after the pebble strikes the water. The area of the circle is given by the function $A(r) = \pi r^2$. Find and interpret $(A \circ r)(t)$.

Synthesis

True or False? **In Exercises 61 and 62, determine whether the statement is true or false. Justify your answer.**

61. If $f(x) = x + 1$ and $g(x) = 6x$, then $(f \circ g)(x) = (g \circ f)(x)$.

62. If you are given two functions $f(x)$ and $g(x)$, you can calculate $(f \circ g)(x)$ if and only if the range of g is a subset of the domain of f.

63. ***Think About It*** You are a sales representative for an automobile manufacturer. You are paid an annual salary, plus a bonus of 3% of your sales over \$500,000. Consider the two functions

$$f(x) = x - 500{,}000 \quad \text{and} \quad g(x) = 0.03x.$$

If x is greater than \$500,000, which of the following represents your bonus? Explain your reasoning.

(a) $f(g(x))$ (b) $g(f(x))$

64. ***Proof*** Prove that the product of two odd functions is an even function, and that the product of two even functions is an even function.

65. ***Conjecture*** Use examples to hypothesize whether the product of an odd function and an even function is even or odd. Then prove your hypothesis.

Review

66. Find the domain of the function.

$$f(x) = \frac{x}{5x + 7}$$

Average Rate of Change **In Exercises 67–70, find the difference quotient**

$$\frac{f(x + h) - f(x)}{h}$$

and simplify your answer.

67. $f(x) = 3x - 4$

68. $f(x) = 1 - x^2$

69. $f(x) = \dfrac{4}{x}$

70. $f(x) = \sqrt{2x + 1}$

In Exercises 71–74, find an equation of the line that passes through the given point and has the indicated slope. Sketch the line.

71. $(2, -4), m = 3$

72. $(-6, 3), m = -1$

73. $(8, -1), m = -\frac{3}{2}$

74. $(7, 0), m = \frac{5}{7}$

2.7 Inverse Functions

▶ What you should learn

- How to find inverse functions informally and verify that two functions are inverse functions of each other
- How to use graphs of functions to determine whether functions have inverse functions
- How to use the Horizontal Line Test to determine if functions are one-to-one
- How to find inverse functions algebraically

▶ Why you should learn it

Inverse functions can be used to model and solve real-life problems. For instance, in Exercise 79 on page 245, an inverse function can be used to determine the year in which there were a given number of households in the United States.

Michelle Bridwell/PhotoEdit

Inverse Functions

Recall from Section 2.2 that a function can be represented by a set of ordered pairs. For instance, the function $f(x) = x + 4$ from the set $A = \{1, 2, 3, 4\}$ to the set $B = \{5, 6, 7, 8\}$ can be written as follows.

$$f(x) = x + 4\text{: } \{(1, 5), (2, 6), (3, 7), (4, 8)\}$$

In this case, by interchanging the first and second coordinates of each of these ordered pairs, you can form the **inverse function** of f, which is denoted by f^{-1}. It is a function from the set B to the set A, and can be written as follows.

$$f^{-1}(x) = x - 4\text{: } \{(5, 1), (6, 2), (7, 3), (8, 4)\}$$

Note that the domain of f is equal to the range of f^{-1}, and vice versa, as shown in Figure 2.63. Also note that the functions f and f^{-1} have the effect of "undoing" each other. In other words, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function.

$$f(f^{-1}(x)) = f(x - 4) = (x - 4) + 4 = x$$

$$f^{-1}(f(x)) = f^{-1}(x + 4) = (x + 4) - 4 = x$$

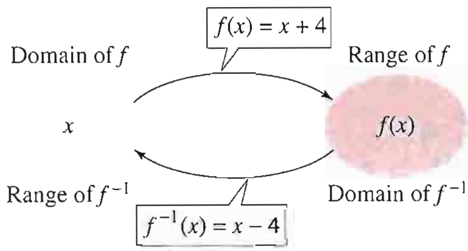


FIGURE 2.63

Example 1 ▶ Finding Inverse Functions Informally

Find the inverse function of $f(x) = 4x$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

Solution

The function f *multiplies* each input by 4. To "undo" this function, you need to *divide* each input by 4. So, the inverse function of $f(x) = 4x$ is

$$f^{-1}(x) = \frac{x}{4}.$$

You can verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function as follows.

$$f(f^{-1}(x)) = f\left(\frac{x}{4}\right) = 4\left(\frac{x}{4}\right) = x \qquad f^{-1}(f(x)) = f^{-1}(4x) = \frac{4x}{4} = x$$

Exploration

Consider the functions

$$f(x) = x + 2$$

and

$$f^{-1}(x) = x - 2.$$

Evaluate $f(f^{-1}(x))$ and $f^{-1}(f(x))$ for the indicated values of x. What can you conclude about the functions?

x	-10	0	7	45
$f(f^{-1}(x))$				
$f^{-1}(f(x))$				

Definition of Inverse Function

Let f and g be two functions such that

$$f(g(x)) = x \qquad \text{for every } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \qquad \text{for every } x \text{ in the domain of } f.$$

Under these conditions, the function g is the **inverse function** of the function f. The function g is denoted by f^{-1} (read "f-inverse"). So,

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

The domain of f must be equal to the range of f^{-1}, and the range of f must be equal to the domain of f^{-1}.

Don't be confused by the use of -1 to denote the inverse function f^{-1}. In this text, whenever f^{-1} is written, it *always* refers to the inverse function of the function f and *not* to the reciprocal of $f(x)$.

If the function g is the inverse function of the function f, it must also be true that the function f is the inverse function of the function g. For this reason, you can say that the functions f and g are *inverse functions of each other*.

Example 2 ▶ Verifying Inverse Functions

Which of the functions is the inverse function of $f(x) = \dfrac{5}{x-2}$?

$$g(x) = \frac{x-2}{5} \qquad h(x) = \frac{5}{x} + 2$$

Solution

By forming the composition of f with g, you have

$$f(g(x)) = f\left(\frac{x-2}{5}\right)$$

$$= \frac{5}{\left(\dfrac{x-2}{5}\right) - 2} \qquad \text{Substitute } \frac{x-2}{5} \text{ for } x.$$

$$= \frac{25}{x-12} \neq x.$$

Because this composition is not equal to the identity function x, it follows that g *is not* the inverse function of f. By forming the composition of f with h, you have

$$f(h(x)) = f\left(\frac{5}{x} + 2\right) = \frac{5}{\left(\dfrac{5}{x} + 2\right) - 2} = \frac{5}{\left(\dfrac{5}{x}\right)} = x.$$

So, it appears that h *is* the inverse function of f. You can confirm this by showing that the composition of h with f is also equal to the identity function.

The Graph of an Inverse Function

The graphs of a function f and its inverse function f^{-1} are related to each other in the following way. If the point (a, b) lies on the graph of f, then the point (b, a) must lie on the graph of f^{-1}, and vice versa. This means that the graph of f^{-1} is a *reflection* of the graph of f in the line $y = x$, as shown in Figure 2.64.

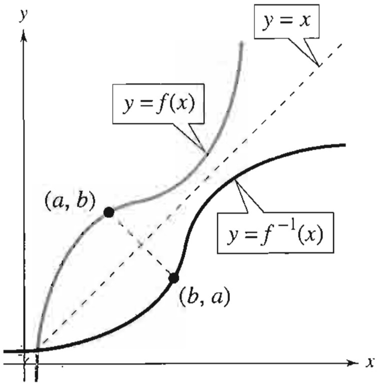


FIGURE 2.64

Example 3 ▶ The Graphs of f and f^{-1}

Sketch the graphs of the inverse functions $f(x) = 2x - 3$ and $f^{-1}(x) = \frac{1}{2}(x + 3)$ on the same rectangular coordinate system and show that the graphs are reflections of each other in the line $y = x$.

Solution

The graphs of f and f^{-1} are shown in Figure 2.65. It appears that the graphs are reflections of each other in the line $y = x$. You can further verify this reflective property by testing a few points on each graph. Note in the following list that if the point (a, b) is on the graph of f, the point (b, a) is on the graph of f^{-1}.

Graph of $f(x) = 2x - 3$	*Graph of* $f^{-1}(x) = \frac{1}{2}(x + 3)$
$(-1, -5)$	$(-5, -1)$
$(0, -3)$	$(-3, 0)$
$(1, -1)$	$(-1, 1)$
$(2, 1)$	$(1, 2)$
$(3, 3)$	$(3, 3)$

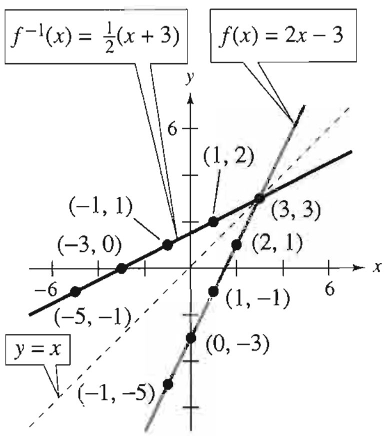


FIGURE 2.65

Example 4 ▶ Finding Inverse Functions Graphically

Sketch the graphs of the inverse functions $f(x) = x^2 (x \geq 0)$ and $f^{-1}(x) = \sqrt{x}$ on the same rectangular coordinate system and show that the graphs are reflections of each other in the line $y = x$.

Solution

The graphs of f and f^{-1} are shown in Figure 2.66. It appears that the graphs are reflections of each other in the line $y = x$. You can further verify this reflective property by testing a few points on each graph. Note in the following list that if the point (a, b) is on the graph of f, the point (b, a) is on the graph of f^{-1}.

Graph of $f(x) = x^2, \quad x \geq 0$	*Graph of* $f^{-1}(x) = \sqrt{x}$
$(0, 0)$	$(0, 0)$
$(1, 1)$	$(1, 1)$
$(2, 4)$	$(4, 2)$
$(3, 9)$	$(9, 3)$

Try showing that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

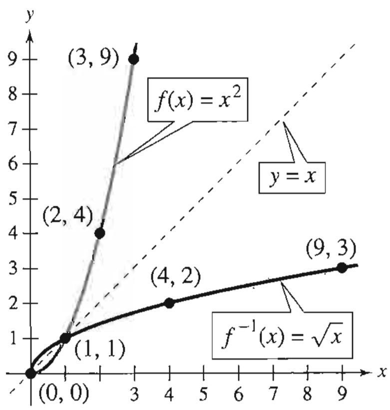


FIGURE 2.66

One-to-One Functions

The reflective property of the graphs of inverse functions gives you a nice *geometric* test for determining whether a function has an inverse function. This test is called the **Horizontal Line Test** for inverse functions.

> **Horizontal Line Test for Inverse Functions**
>
> A function f has an inverse function if and only if no *horizontal* line intersects the graph of f at more than one point.

If no horizontal line intersects the graph of f at more than one point, then no y-value is matched with more than one x-value. This is the essential characteristic of what are called **one-to-one** functions.

> **One-to-One Functions**
>
> A function f is **one-to-one** if each value of the dependent variable corresponds to exactly one value of the independent variable. A function f has an inverse function if and only if f is one-to-one.

Consider the function $f(x) = x^2$. The table on the left is a table of values for $f(x) = x^2$. The table of values on the right is made up by interchanging the columns of the first table. The table on the right does not represent a function because the input $x = 4$ is matched with two different outputs: $y = -2$ and $y = 2$. So, $f(x) = x^2$ is not one-to-one and does not have an inverse function.

x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4
3	9

x	y
4	-2
1	-1
0	0
1	1
4	2
9	3

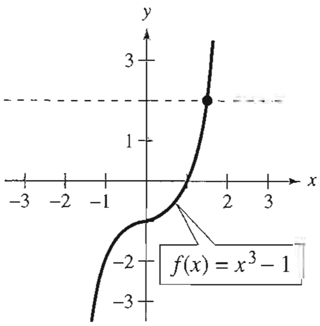


FIGURE 2.67

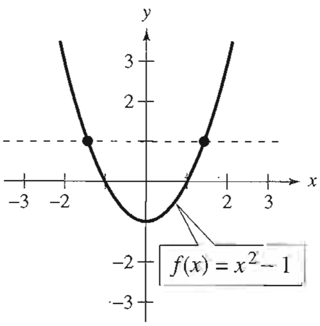


FIGURE 2.68

Example 5 ▶ Applying the Horizontal Line Test

a. The graph of the function $f(x) = x^3 - 1$ is shown in Figure 2.67. Because no horizontal line intersects the graph of f at more than one point, you can conclude that f *is* a one-to-one function and *does* have an inverse function.

b. The graph of the function $f(x) = x^2 - 1$ is shown in Figure 2.68. Because it is possible to find a horizontal line that intersects the graph of f at more than one point, you can conclude that f *is not* a one-to-one function and *does not* have an inverse function.

Finding Inverse Functions Algebraically

For simple functions (such as the one in Example 1), you can find inverse functions by inspection. For more complicated functions, however, it is best to use the following guidelines. The key step in these guidelines is Step 3—interchanging the roles of x and y. This step corresponds to the fact that inverse functions have ordered pairs with the coordinates reversed.

Finding an Inverse Function

1. Use the Horizontal Line Test to decide whether f has an inverse function.
2. In the equation for $f(x)$, replace $f(x)$ by y.
3. Interchange the roles of x and y, and solve for y.
4. Replace y by $f^{-1}(x)$ in the new equation.
5. Verify that f and f^{-1} are inverse functions of each other by showing that the domain of f is equal to the range of f^{-1}, the range of f is equal to the domain of f^{-1}, and $f(f^{-1}(x)) = x = f^{-1}(f(x))$.

STUDY TIP

Note what happens when you try to find the inverse function of a function that is not one-to-one.

$f(x) = x^2 + 1$ Original function

$y = x^2 + 1$ Replace $f(x)$ by y.

$x = y^2 + 1$ Interchange x and y.

$x - 1 = y^2$ Isolate y-term.

$y = \pm\sqrt{x - 1}$ Solve for y.

You obtain two y-values for each x.

Example 6 ▶ Finding an Inverse Function Algebraically

Find the inverse function of

$$f(x) = \frac{5 - 3x}{2}.$$

Solution

The graph of f is a line, as shown in Figure 2.69. This graph passes the Horizontal Line Test. So, you know that f is one-to-one and has an inverse function.

$f(x) = \dfrac{5 - 3x}{2}$ Write original function.

$y = \dfrac{5 - 3x}{2}$ Replace $f(x)$ by y.

$x = \dfrac{5 - 3y}{2}$ Interchange x and y.

$2x = 5 - 3y$ Multiply each side by 2.

$3y = 5 - 2x$ Isolate the y-term.

$y = \dfrac{5 - 2x}{3}$ Solve for y.

$f^{-1}(x) = \dfrac{5 - 2x}{3}$ Replace y by $f^{-1}(x)$.

Note that both f and f^{-1} have domains and ranges that consist of the entire set of real numbers. Check that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Exploration

Restrict the domain of $f(x) = x^2 + 1$ to $x \geq 0$. Use a graphing utility to graph the function. Does the restricted function have an inverse function? Explain.

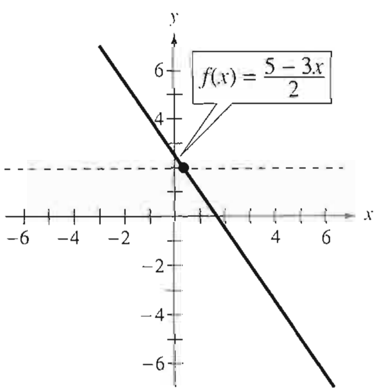


FIGURE 2.69

Example 7 ▶ Finding an Inverse Function

Find the inverse function of

$$f(x) = \sqrt[3]{x+1}.$$

Solution

The graph of f is a curve, as shown in Figure 2.70. Because this graph passes the Horizontal Line Test, you know that f is one-to-one and has an inverse function.

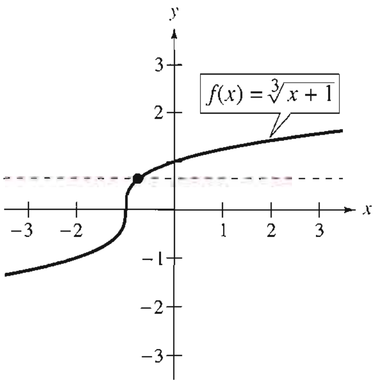


FIGURE 2.70

$f(x) = \sqrt[3]{x+1}$	Write original function.
$y = \sqrt[3]{x+1}$	Replace $f(x)$ by y.
$x = \sqrt[3]{y+1}$	Interchange x and y.
$x^3 = y + 1$	Cube each side.
$x^3 - 1 = y$	Solve for y.
$x^3 - 1 = f^{-1}(x)$	Replace y by $f^{-1}(x)$.

Both f and f^{-1} have domains and ranges that consist of the entire set of real numbers. You can verify this result numerically as shown in the tables below.

x	$f(x)$
-28	-3
-9	-2
-2	-1
-1	0
0	1
7	2
26	3

x	$f^{-1}(x)$
-3	-28
-2	-9
-1	-2
0	-1
1	0
2	7
3	26

Writing ABOUT MATHEMATICS

The Existence of an Inverse Function Write a short paragraph describing why the following functions do or do not have inverse functions.

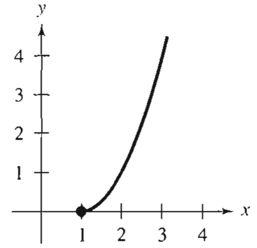
a. Let x represent the retail price of an item (in dollars), and let $f(x)$ represent the sales tax on the item. Assume that the sales tax is 6% of the retail price *and* that the sales tax is rounded to the nearest cent. Does this function have an inverse function? (*Hint:* Can you undo this function? For instance, if you know that the sales tax is \$0.12, can you determine exactly what the retail price is?)

b. Let x represent the temperature in degrees Celsius, and let $f(x)$ represent the temperature in degrees Fahrenheit. Does this function have an inverse function? (*Hint:* The formula for converting from degrees Celsius to degrees Fahrenheit is $F = \frac{9}{5}C + 32$.)

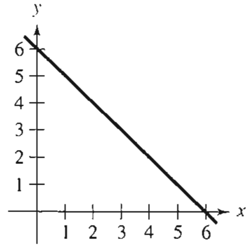
2.7 Exercises

In Exercises 1–4, match the graph of the function with the graph of its inverse function. [The graphs of the inverse functions are labeled (a), (b), (c), and (d).]

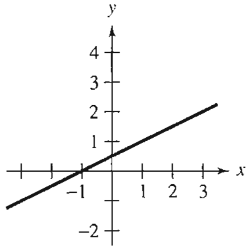
(a)



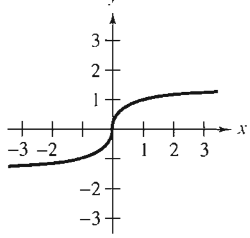
(b)



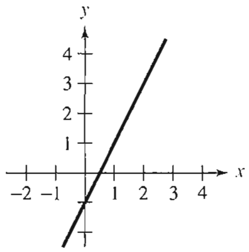
(c)



(d)



1.



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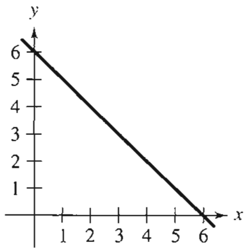
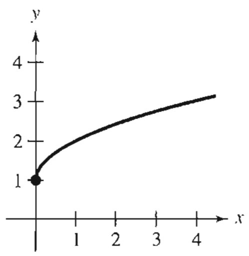


3.



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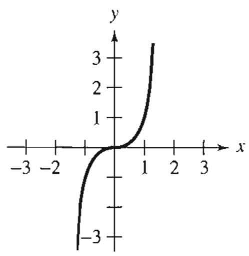


In Exercises 5–12, find the inverse function of f informally. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

5. $f(x) = 6x$ **6.** $f(x) = \frac{1}{3}x$

7. $f(x) = x + 9$ **8.** $f(x) = x - 4$

9. $f(x) = 3x + 1$ **10.** $f(x) = \dfrac{x-1}{5}$

11. $f(x) = \sqrt[3]{x}$ **12.** $f(x) = x^5$

In Exercises 13–24, show that f and g are inverse functions (a) algebraically and (b) graphically.

13. $f(x) = 2x,\quad g(x) = \dfrac{x}{2}$

14. $f(x) = x - 5,\quad g(x) = x + 5$

15. $f(x) = 7x + 1,\quad g(x) = \dfrac{x-1}{7}$

16. $f(x) = 3 - 4x,\quad g(x) = \dfrac{3-x}{4}$

17. $f(x) = \dfrac{x^3}{8},\quad g(x) = \sqrt[3]{8x}$

18. $f(x) = \dfrac{1}{x},\quad g(x) = \dfrac{1}{x}$

19. $f(x) = \sqrt{x-4},\quad g(x) = x^2 + 4,\quad x \ge 0$

20. $f(x) = 1 - x^3,\quad g(x) = \sqrt[3]{1-x}$

21. $f(x) = 9 - x^2,\quad x \ge 0,\quad g(x) = \sqrt{9-x},\quad x \le 9$

22. $f(x) = \dfrac{1}{1+x},\quad x \ge 0$

$g(x) = \dfrac{1-x}{x},\quad 0 < x \le 1$

23. $f(x) = \dfrac{x-1}{x+5},\quad g(x) = -\dfrac{5x+1}{x-1}$

24. $f(x) = \dfrac{x+3}{x-2},\quad g(x) = \dfrac{2x+3}{x-1}$

In Exercises 25 and 26, does the function have an inverse function?

25.

x	$f(x)$
−1	−2
0	1
1	2
2	1
3	−2
4	−6

26.

x	$f(x)$
−3	10
−2	6
−1	4
0	1
2	−3
3	−10

In Exercises 27 and 28, use the table of values for $y = f(x)$ to complete a table for $y = f^{-1}(x)$.

27.

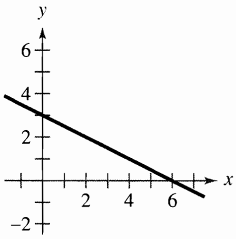
x	-2	-1	0	1	2	3
$f(x)$	-2	0	2	4	6	8

28.

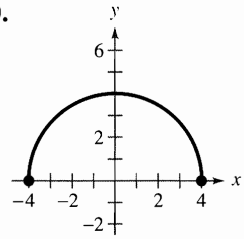
x	-3	-2	-1	0	1	2
$f(x)$	-10	-7	-4	-1	2	5

In Exercises 29–32, does the function have an inverse function?

29.



30.



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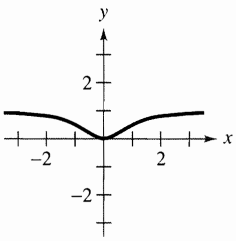

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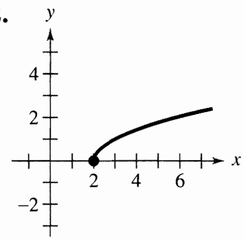


In Exercises 33–38, use a graphing utility to graph the function, and use the Horizontal Line Test to determine whether the function is one-to-one and so has an inverse function.

33. $g(x) = \dfrac{4 - x}{6}$

34. $f(x) = 10$

35. $h(x) = |x + 4| - |x - 4|$

36. $g(x) = (x + 5)^3$

37. $f(x) = -2x\sqrt{16 - x^2}$

38. $f(x) = \frac{1}{8}(x + 2)^2 - 1$

In Exercises 39–54, find the inverse function of f. Then graph both f and f^{-1} on the same set of coordinate axes.

39. $f(x) = 2x - 3$ **40.** $f(x) = 3x + 1$

41. $f(x) = x^5 - 2$ **42.** $f(x) = x^3 + 1$

43. $f(x) = \sqrt{x}$ **44.** $f(x) = x^2, \quad x \ge 0$

45. $f(x) = \sqrt{4 - x^2}, \quad 0 \le x \le 2$

46. $f(x) = x^2 - 2, \quad x \le 0$

47. $f(x) = \dfrac{4}{x}$ **48.** $f(x) = -\dfrac{2}{x}$

49. $f(x) = \dfrac{x + 1}{x - 2}$ **50.** $f(x) = \dfrac{x - 3}{x + 2}$

51. $f(x) = \sqrt[3]{x - 1}$ **52.** $f(x) = x^{3/5}$

53. $f(x) = \dfrac{6x + 4}{4x + 5}$ **54.** $f(x) = \dfrac{8x - 4}{2x + 6}$

In Exercises 55–68, determine whether the function has an inverse function. If it does, find the inverse function.

55. $f(x) = x^4$ **56.** $f(x) = \dfrac{1}{x^2}$

57. $g(x) = \dfrac{x}{8}$ **58.** $f(x) = 3x + 5$

59. $p(x) = -4$ **60.** $f(x) = \dfrac{3x + 4}{5}$

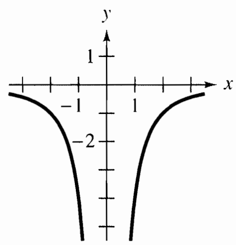
61. $f(x) = (x + 3)^2, \quad x \ge -3$

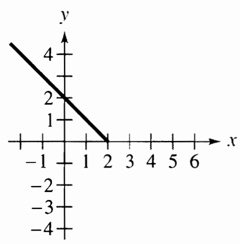
62. $q(x) = (x - 5)^2$

63. $f(x) = \begin{cases} x + 3, & x < 0 \\ 6 - x, & x \ge 0 \end{cases}$

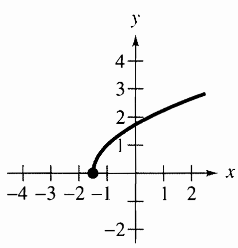
64. $f(x) = \begin{cases} -x, & x \le 0 \\ x^2 - 3x, & x > 0 \end{cases}$

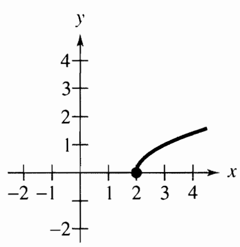
65. $h(x) = -\dfrac{4}{x^2}$ **66.** $f(x) = |x - 2|, \quad x \le 2$





67. $f(x) = \sqrt{2x + 3}$ **68.** $f(x) = \sqrt{x - 2}$





In Exercises 69–74, use the functions $f(x) = \frac{1}{8}x - 3$ and $g(x) = x^3$ to find the indicated value or function.

69. $(f^{-1} \circ g^{-1})(1)$

70. $(g^{-1} \circ f^{-1})(-3)$

71. $(f^{-1} \circ f^{-1})(6)$

72. $(g^{-1} \circ g^{-1})(-4)$

73. $(f \circ g)^{-1}$

74. $g^{-1} \circ f^{-1}$

In Exercises 75–78, use the functions $f(x) = x + 4$ and $g(x) = 2x - 5$ to find the specified function.

75. $g^{-1} \circ f^{-1}$

76. $f^{-1} \circ g^{-1}$

77. $(f \circ g)^{-1}$

78. $(g \circ f)^{-1}$

▶ Model It

79. ***U.S. Households*** The number of households f (in thousands) in the United States from 1994 to 2000 are shown in the table. The time (in years) is given by t, with $t = 4$ corresponding to 1994. (Source: U.S. Census Bureau)

Year, t	Households, $f(t)$
4	97,107
5	98,990
6	99,627
7	101,018
8	102,528
9	103,874
10	104,705

(a) Find $f^{-1}(103{,}874)$.

(b) What does f^{-1} mean in the context of the problem?

(c) Use the *regression* feature of a graphing utility to find a linear model for the data, $y = mx + b$. (Round m and b to two decimal places.)

(d) Algebraically find the inverse function of the linear model in part (c).

(e) Use the inverse function of the linear model you found in part (d) to approximate $f^{-1}(111, 254)$.

80. ***Bottled Water Consumption*** The per capita consumption f (in gallons) of bottled water in the United States from 1994 through 1999 is shown in the table. The time (in years) is given by t, with $t = 4$ corresponding to 1994. (Source: U.S. Department of Agriculture)

t	$f(t)$
4	10.7
5	11.6
6	12.5
7	13.1
8	16.0
9	18.1

(a) Does f^{-1} exist?

(b) If f^{-1} exists, what does it represent in the context of the problem?

(c) If f^{-1} exists, find $f^{-1}(16.0)$.

81. ***Miles Traveled*** The total number f (in billions) of miles traveled by motor vehicles in the United States from 1992 through 1999 is shown in the table below. The time (in years) is given by t, with $t = 2$ corresponding to 1992. (Source: U.S. Federal Highway Administration)

Year, t	Miles traveled, $f(t)$
2	2247
3	2296
4	2358
5	2423
6	2486
7	2562
8	2632
9	2691

(a) Does f^{-1} exist?

(b) If f^{-1} exists, what does it mean in the context of the problem?

(c) If f^{-1} exists, find $f^{-1}(2632)$.

(d) If the table was extended to 2000 and if the total number of miles traveled by motor vehicles for that year was 2423 billion, would f^{-1} exist? Explain.

82. ***Hourly Wage*** Your wage is \$8.00 per hour plus \$0.75 for each unit produced per hour. So, your hourly wage y in terms of the number of units produced is $y = 8 + 0.75x$.

(a) Find the inverse function.

(b) What does each variable represent in the inverse function?

(c) Determine the number of units produced when your hourly wage is \$22.25.

83. ***Diesel Mechanics*** The function

$$y = 0.03x^2 + 245.50, \qquad 0 < x < 100$$

approximates the exhaust temperature y in degrees Fahrenheit, where x is the percent load for a diesel engine.

(a) Find the inverse function. What does each variable represent in the inverse function?

(b) Use a graphing utility to graph the inverse function.

(c) The exhaust temperature of the engine must not exceed 500 degrees Fahrenheit. What is the percent load interval?

84. ***Cost*** You need a total of 50 pounds of two types of ground beef costing \$1.25 and \$1.60 per pound, respectively. A model for the total cost y of the two types of beef is

$$y = 1.25x + 1.60(50 - x)$$

where x is the number of pounds of the less expensive ground beef.

(a) Find the inverse function of the cost function. What does each variable represent in the inverse function?

(b) Use the context of the problem to determine the domain of the inverse function.

(c) Determine the number of pounds of the less expensive ground beef purchased when the total cost is \$73.

Synthesis

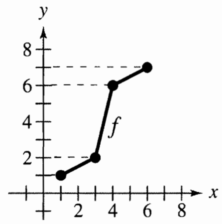
True or False? **In Exercises 85 and 86, determine whether the statement is true or false. Justify your answer.**

85. If f is an even function, f^{-1} exists.

86. If the inverse function of f exists and the graph of f has a y-intercept, the y-intercept of f is an x-intercept of f^{-1}.

In Exercises 87–90, use the graph of the function f to create a table of values for the given points. Then create a second table that can be used to find f^{-1}, and sketch the graph of f^{-1} if possible.

87.



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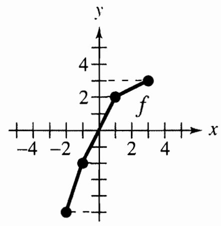

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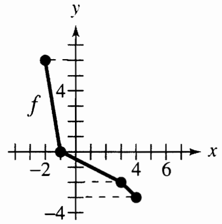

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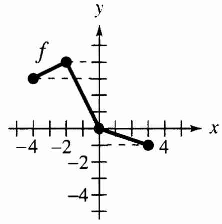


91. ***Think About It*** The function

$$f(x) = k(2 - x - x^3)$$

has an inverse function, and $f^{-1}(3) = -2$. Find k.

92. ***Think About It*** The function

$$f(x) = k(x^3 + 3x - 4)$$

has an inverse function, and $f^{-1}(-5) = 2$. Find k.

Review

In Exercises 93–100, solve the equation by any convenient method.

93. $x^2 = 64$

94. $(x - 5)^2 = 8$

95. $4x^2 - 12x + 9 = 0$

96. $9x^2 + 12x + 3 = 0$

97. $x^2 - 6x + 4 = 0$

98. $2x^2 - 4x - 6 = 0$

99. $50 + 5x = 3x^2$

100. $2x^2 + 4x - 9 = 2(x - 1)^2$

101. Find two consecutive positive even integers whose product is 288.

102. ***Geometry*** A triangular sign has a height that is twice its base. The area of the sign is 10 square feet. Find the base and height of the sign

Chapter Summary

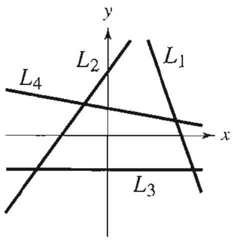
▶ *What* did you learn?

Section 2.1	Review Exercises
☐ How to use slope to graph linear equations in two variables	1–14
☐ How to find slopes of lines	15–18
☐ How to write linear equations in two variables	19–26
☐ How to use slope to identify parallel and perpendicular lines	27, 28
☐ How to use linear equations in two variables to model and solve real-life problems	29–32
Section 2.2	
☐ How to determine whether relations between two variables are functions	33–38
☐ How to use function notation and evaluate functions	39–42
☐ How to find the domains of functions	43–48
☐ How to find the difference quotients	49, 50
☐ How to use functions to model and solve real-life problems	51–54
Section 2.3	
☐ How to use the Vertical Line Test for functions	55–58
☐ How to find the zeros of functions	59–62
☐ How to determine intervals on which functions are increasing or decreasing	63, 64
☐ How to identify even and odd functions	65–68
Section 2.4	
☐ How to identify and graph linear, squaring, cubic, square root, reciprocal, step, and other piecewise-defined functions	69–82
☐ How to recognize graphs of common functions	83, 84
Section 2.5	
☐ How to use vertical and horizontal shifts to sketch graphs of functions	85–88
☐ How to use reflections to sketch graphs of functions	89–94
☐ How to use nonrigid transformations to sketch graphs of functions	95–98
Section 2.6	
☐ How to add, subtract, multiply, and divide functions	99, 100
☐ How to find the composition of one function with another function	101–104
☐ How to use combinations of functions to model and solve real-life problems	105, 106
Section 2.7	
☐ How to find inverse functions informally and verify that two functions are inverse functions of each other	107, 108
☐ How to use graphs to determine whether functions have inverse functions	109, 110
☐ How to use the Horizontal Line Test to determine if functions are one-to-one	111–114
☐ How to find inverse functions algebraically	115–120

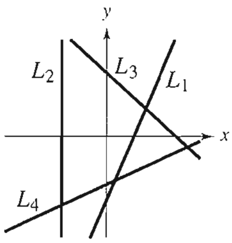
Review Exercises

2.1 **In Exercises 1 and 2, identify the line that has each slope.**

1. (a) $m = \frac{3}{2}$
   (b) $m = 0$
   (c) $m = -3$
   (d) $m = -\frac{1}{5}$



2. (a) m is undefined.
   (b) $m = -1$
   (c) $m = \frac{5}{2}$
   (d) $m = \frac{1}{2}$



In Exercises 3–10, sketch the graph of the linear equation.

3. $y = -2x - 7$
4. $y = 4x - 3$
5. $y = 6$
6. $x = -3$
7. $y = 3x + 13$
8. $y = -10x + 9$
9. $y = -\frac{5}{2}x - 1$
10. $y = \frac{5}{6}x + 5$

In Exercises 11 and 12, use the concept of slope to find t such that the three points are on the same line.

11. $(-2, 5), (0, t), (1, 1)$
12. $(-6, 1), (1, t), (10, 5)$

In Exercises 13 and 14, use the point on the line and the slope of the line to find three additional points through which the line passes. (There are many correct answers.)

	Point	*Slope*
13.	$(2, -1)$	$m = \frac{1}{4}$
14.	$(-3, 5)$	$m = -\frac{3}{2}$

In Exercises 15–18, plot the points and find the slope of the line passing through the points.

15. $(3, -4), (-7, 1)$
16. $(-1, 8), (6, 5)$
17. $(-4.5, 6), (2.1, 3)$
18. $(-3, 2), (8, 2)$

In Exercises 19–22, find an equation of the line that passes through the points.

19. $(0, 0), (0, 10)$
20. $(2, 5), (-2, -1)$
21. $(-1, 4), (2, 0)$
22. $(11, -2), (6, -1)$

In Exercises 23–26, find an equation of the line that passes through the given point and has the specified slope. Sketch the line.

	Point	*Slope*
23.	$(0, -5)$	$m = \frac{3}{2}$
24.	$(-2, 6)$	$m = 0$
25.	$(10, -3)$	$m = -\frac{1}{2}$
26.	$(-8, 5)$	m is undefined.

In Exercises 27 and 28, write an equation of the line through the point (a) parallel to the given line and (b) perpendicular to the given line.

	Point	*Line*
27.	$(3, -2)$	$5x - 4y = 8$
28.	$(-8, 3)$	$2x + 3y = 5$

Rate of Change **In Exercises 29 and 30, you are given the dollar value of a product in the year 2004 *and* the rate at which the value of the item is expected to change during the next 5 years. Write a linear equation that gives the dollar value V of the product in terms of the year t. (Let $t = 4$ represent 2004.)**

	2004 Value	*Rate*
29.	\$12,500	\$850 increase per year
30.	\$72.95	\$5.15 increase per year

31. ***Sales*** During the second and third quarters of the year, a salvage yard had sales of \$160,000 and \$185,000, respectively. The growth of sales follows a linear pattern. Estimate sales during the fourth quarter.

32. ***Inflation*** The dollar value of a product in 2005 is \$85, and the product is expected to increase in value at a rate of \$3.75 per year.
   (a) Write a linear equation that gives the dollar value V of the product in terms of the year t. (Let $t = 5$ represent 2005.)
   (b) Use a graphing utility to graph the equation found in part (a).
   (c) Move the cursor along the graph of the sales model to estimate the dollar value of the product in 2010.

2.2 In Exercises 33 and 34, determine which of the sets of ordered pairs represents a function from *A* to *B*. Give reasons for your answers.

33. $A = \{10, 20, 30, 40\}$ and $B = \{0, 2, 4, 6\}$
   (a) $\{(20, 4), (40, 0), (20, 6), (30, 2)\}$
   (b) $\{(10, 4), (20, 4), (30, 4), (40, 4)\}$
   (c) $\{(40, 0), (30, 2), (20, 4), (10, 6)\}$
   (d) $\{(20, 2), (10, 0), (40, 4)\}$

34. $A = \{u, v, w\}$ and $B = \{-2, -1, 0, 1, 2\}$
   (a) $\{(v, -1), (u, 2), (w, 0), (u, -2)\}$
   (b) $\{(u, -2), (v, 2), (w, 1)\}$
   (c) $\{(u, 2), (v, 2), (w, 1), (w, 1)\}$
   (d) $\{(w, -2), (v, 0), (w, 2)\}$

In Exercises 35–38, determine whether the equation represents *y* as a function of *x*.

35. $16x - y^4 = 0$ 36. $2x - y - 3 = 0$
37. $y = \sqrt{1 - x}$ 38. $|y| = x + 2$

In Exercises 39–42, evaluate the function as indicated. Simplify your answers.

39. $f(x) = x^2 + 1$
   (a) $f(2)$ (b) $f(-4)$ (c) $f(t^2)$ (d) $f(t + 1)$

40. $g(x) = x^{4/3}$
   (a) $g(8)$ (b) $g(t + 1)$ (c) $g(-27)$ (d) $g(-x)$

41. $h(x) = \begin{cases} 2x + 1, & x \le -1 \\ x^2 + 2, & x > -1 \end{cases}$
   (a) $h(-2)$ (b) $h(-1)$ (c) $h(0)$ (d) $h(2)$

42. $f(x) = \dfrac{4}{x^2 + 1}$
   (a) $f(1)$ (b) $f(-5)$ (c) $f(-t)$ (d) $f(0)$

In Exercises 43–48, determine the domain of the function. Verify your result with a graph.

43. $f(x) = \sqrt{25 - x^2}$ 44. $f(x) = 3x + 4$
45. $g(s) = \dfrac{5}{3s - 9}$ 46. $f(x) = \sqrt{x^2 + 8x}$
47. $h(x) = \dfrac{x}{x^2 - x - 6}$ 48. $h(t) = |t + 1|$

In Exercises 49 and 50, find the difference quotient and simplify your answer.

49. $f(x) = 2x^2 + 3x - 1$, $\dfrac{f(x + h) - f(x)}{h}$, $h \neq 0$

50. $f(x) = x^3 - 5x^2 + x$, $\dfrac{f(x + h) - f(x)}{h}$, $h \neq 0$

51. ***Physics*** The velocity of a ball thrown vertically upward from ground level is $v(t) = -32t + 48$, where t is the time in seconds and v is the velocity in feet per second.
   (a) Find the velocity when $t = 1$.
   (b) Find the time when the ball reaches its maximum height. [*Hint:* Find the time when $v(t) = 0$.]
   (c) Find the velocity when $t = 2$.

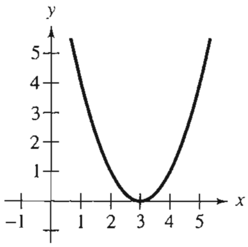
52. ***Total Cost*** A hand tool manufacturer produces a product for which the variable cost is \$5.35 per unit and the fixed costs are \$16,000. The company sells the product for \$8.20 and can sell all that it produces.
   (a) Find the total cost as a function of x, the number of units produced.
   (b) Find the profit as a function of x.

53. ***Geometry*** A wire 24 inches long is to be cut into four pieces to form a rectangle with one side of length x.
   (a) Write the area A of the rectangle as a function of x.
   (b) Determine the domain of the function.

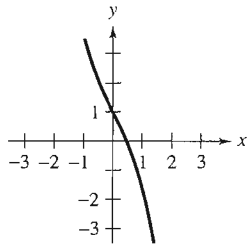
54. ***Mixture Problem*** From a full 50-liter container of a 40% concentration of acid, x liters is removed and replaced with 100% acid.
   (a) Write the amount of acid in the final mixture as a function of x.
   (b) Determine the domain and range of the function.
   (c) Determine x if the final mixture is 50% acid.

2.3 **In Exercises 55–58, use the Vertical Line Test to determine whether y is a function of x. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

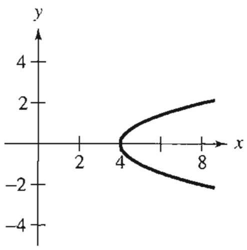
55. $y = (x - 3)^2$



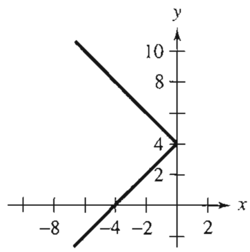
56. $y = -\frac{3}{5}x^3 - 2x + 1$



57. $x - 4 = y^2$



58. $x = -|4 - y|$



In Exercises 59–62, find the zeros of the function.

59. $f(x) = 3x^2 - 16x + 21$

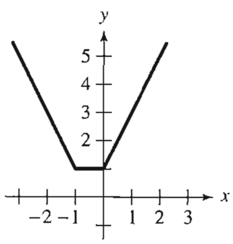
60. $f(x) = 5x^2 + 4x - 1$

61. $f(x) = \dfrac{8x + 3}{11 - x}$

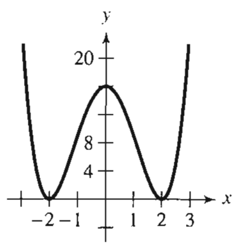
62. $f(x) = x^3 - x^2 - 25x + 25$

In Exercises 63 and 64, determine the intervals over which the function is increasing, decreasing, or constant.

63. $f(x) = |x| + |x + 1|$



64. $f(x) = (x^2 - 4)^2$



In Exercises 65–68, determine whether the function is even, odd, or neither.

65. $f(x) = x^5 + 4x - 7$

66. $f(x) = x^4 - 20x^2$

67. $f(x) = 2x\sqrt{x^2 + 3}$

68. $f(x) = \sqrt[5]{6x^2}$

2.4 **In Exercises 69–72, write the linear function f such that it has the indicated function values. Sketch a graph of the function.**

69. $f(2) = -6,\ f(-1) = 3$

70. $f(0) = -5,\ f(4) = -8$

71. $f\left(-\frac{4}{5}\right) = 2,\ f\left(\frac{11}{5}\right) = 7$

72. $f(3.3) = 5.6,\ f(-4.7) = -1.4$

In Exercises 73–82, graph the function.

73. $f(x) = 3 - x^2$

74. $h(x) = x^3 - 2$

75. $f(x) = -\sqrt{x}$

76. $f(x) = \sqrt{x + 1}$

77. $g(x) = \dfrac{3}{x}$

78. $g(x) = \dfrac{1}{x + 5}$

79. $f(x) = [\![x]\!] - 2$

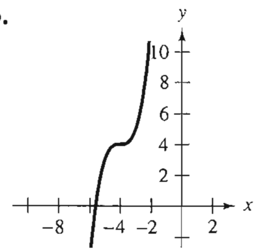
80. $g(x) = [\![x + 4]\!]$

81. $f(x) = \begin{cases} 5x - 3, & x \geq -1 \\ -4x + 5, & x < -1 \end{cases}$

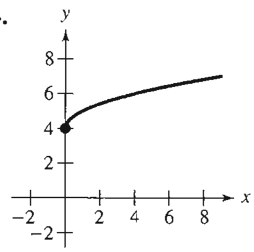
82. $f(x) = \begin{cases} x^2 - 2, & x < -2 \\ 5, & -2 \leq x \leq 0 \\ 8x - 5, & x > 0 \end{cases}$

In Exercises 83 and 84, identify the transformed common function shown in the graph.

83.



84.



2.5 **In Exercises 85–98, identify the transformation of the graph of f and sketch the graph of h.**

85. $f(x) = x^2,\quad h(x) = x^2 - 9$

86. $f(x) = x^3,\quad h(x) = (x - 2)^3 + 2$

87. $f(x) = \sqrt{x},\quad h(x) = \sqrt{x - 7}$

88. $f(x) = |x|,\quad h(x) = |x + 3| - 5$

89. $f(x) = x^2,\quad h(x) = -(x + 3)^2 + 1$

90. $f(x) = x^3,\quad h(x) = -(x - 5)^3 - 5$

91. $f(x) = [\![x]\!],\quad h(x) = -[\![x]\!] + 6$

92. $f(x) = \sqrt{x},\quad h(x) = -\sqrt{x + 1} + 9$

93. $f(x) = |x|,\quad h(x) = -|-x + 4| + 6$

94. $f(x) = x^2,\quad h(x) = -(x + 1)^2 - 3$

95. $f(x) = [\![x]\!]$, $h(x) = 5[\![x - 9]\!]$

96. $f(x) = x^3$, $h(x) = -\frac{1}{3}x^3$

97. $f(x) = \sqrt{x}$, $h(x) = -2\sqrt{x - 4}$

98. $f(x) = |x|$, $h(x) = \frac{1}{2}|x| - 1$

2.6 In Exercises 99 and 100, find (a) $(f + g)(x)$, (b) $(f - g)(x)$, (c) $(fg)(x)$, and (d) $(f/g)(x)$. What is the domain of f/g?

99. $f(x) = x^2 + 3$, $g(x) = 2x - 1$

100. $f(x) = x^2 - 4$, $g(x) = \sqrt{3 - x}$

In Exercises 101 and 102, find (a) $f \circ g$ and (b) $g \circ f$. Find the domain of each function and each composite function.

101. $f(x) = \frac{1}{3}x - 3$, $g(x) = 3x + 1$

102. $f(x) = x^3 - 4$, $g(x) = \sqrt[3]{x + 7}$

In Exercise 103 and 104, find two functions f and g such that $(f \circ g)(x) = h(x)$. (There is more than one correct answer.)

103. $h(x) = (6x - 5)^3$

104. $h(x) = \sqrt[3]{x + 2}$

Data Analysis In Exercises 105 and 106, use the table, which shows the total values (in billions of dollars) of U.S. imports from Mexico and Canada for the years 1995 through 1999. The variables y_1 and y_2 represent the total values of imports from Mexico and Canada, respectively. (Source: U.S. Census Bureau)

Year	y_1	y_2
1995	62.1	144.4
1996	74.3	155.9
1997	85.9	168.2
1998	94.6	173.3
1999	109.7	198.3

105. Use a graphing utility to find quadratic models for y_1 and y_2. Let $t = 5$ represent 1995.

106. Use a graphing utility to graph y_1, y_2, and $y_1 + y_2$ in the same viewing window. Use the model to estimate the total value of U.S. imports from Canada and Mexico in 2005.

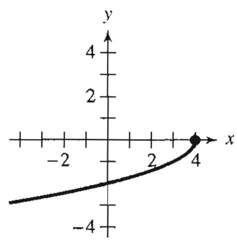
2.7 In Exercises 107 and 108, find the inverse function of f informally. Verify that $f(f^{-1}(x)) = x = f^{-1}(f(x))$.

107. $f(x) = x - 7$

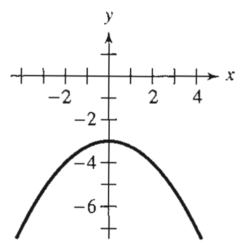
108. $f(x) = x + 5$

In Exercises 109 and 110, determine whether the function has an inverse function.

109.



110.



In Exercises 111–114, use the Horizontal Line Test to determine if the function is one-to-one and so has an inverse function.

111. $f(x) = 4 - \frac{1}{3}x$

112. $f(x) = (x - 1)^2$

113. $h(t) = \dfrac{2}{t - 3}$

114. $g(x) = \sqrt{x + 6}$

In Exercises 115–118, (a) find f^{-1}, (b) sketch the graphs of f and f^{-1} on the same coordinate system, and (c) verify that $f^{-1}(f(x)) = x = f(f^{-1}(x))$.

115. $f(x) = \frac{1}{2}x - 3$

116. $f(x) = 5x - 7$

117. $f(x) = \sqrt{x + 1}$

118. $f(x) = x^3 + 2$

In Exercises 119 and 120, restrict the domain of the function f to an interval over which the function is increasing and determine f^{-1} over that interval.

119. $f(x) = 2(x - 4)^2$

120. $f(x) = |x - 2|$

Synthesis

True or False? In Exercises 121 and 122, determine whether the statement is true or false. Justify your answer.

121. Relative to the graph of $f(x) = \sqrt{x}$, the function $h(x) = -\sqrt{x + 9} - 13$ is shifted 9 units to the left and 13 units downward, then reflected in the x-axis.

122. If f and g are two inverse functions, then the domain of g is equal to the range of f.

123. ***Writing*** Explain how to tell whether a relation between two variables is a function.

124. ***Writing*** Explain the difference between the Vertical Line Test and the Horizontal Line Test.

Chapter Test

The *Interactive* CD-ROM and *Internet* versions of this text offer Chapter Pre-Tests and Chapter Post-Tests, both of which have randomly generated exercises with diagnostic capabilities.

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

In Exercises 1 and 2, find an equation of the line passing through the points. Then sketch the line.

1. $(2, -3), (-4, 9)$ **2.** $(3, 0.8), (7, -6)$

3. Find an equation of the line that passes through the point $(3, 8)$ and is (a) parallel to and (b) perpendicular to the line $-4x + 7y = -5$.

In Exercises 4 and 5, evaluate the function at each specified value.

4. $f(x) = |x + 2| - 15$

(a) $f(-8)$ (b) $f(14)$ (c) $f(x - 6)$

5. $f(x) = \dfrac{\sqrt{x + 9}}{x^2 - 81}$

(a) $f(7)$ (b) $f(-5)$ (c) $f(x - 9)$

In Exercises 6 and 7, determine the domain of the function.

6. $f(x) = \sqrt{100 - x^2}$ **7.** $f(x) = |-x + 6| + 2$

In Exercises 8–10, (a) use a graphing utility to graph the function, (b) approximate the intervals over which the function is increasing, decreasing, or constant, and (c) determine whether the function is even, odd, or neither.

8. $f(x) = 2x^6 + 5x^4 - x^2$ **9.** $f(x) = 4x\sqrt{3 - x}$ **10.** $f(x) = |x + 5|$

11. Sketch the graph of $f(x) = \begin{cases} 3x + 7, & x \le -3 \\ 4x^2 - 1, & x > -3 \end{cases}$.

In Exercises 12–14, identify the common function in the transformation. Then sketch a graph of the function.

12. $h(x) = -[\![x]\!]$ **13.** $h(x) = -\sqrt{x + 5} + 8$ **14.** $h(x) = \frac{1}{4}|x + 1| - 3$

In Exercises 15 and 16, find (a) $(f + g)(x)$, (b) $(f - g)(x)$, (c) $(fg)(x)$, (d) $(f/g)(x)$, (e) $(f \circ g)(x)$, and (f) $(g \circ f)(x)$.

15. $f(x) = 3x^2 - 7, \quad g(x) = -x^2 - 4x + 5$ **16.** $f(x) = \dfrac{1}{x}, \quad g(x) = 2\sqrt{x}$

In Exercises 17–19, determine whether the function has an inverse function, and if so, find the inverse function.

17. $f(x) = x^3 + 8$ **18.** $f(x) = |x^2 - 3| + 6$ **19.** $f(x) = \dfrac{3x\sqrt{x}}{8}$

20. It costs a company \$58 to produce 6 units of a product and \$78 to produce 10 units. How much does it cost to produce 25 units, assuming that the cost function is linear?

Cumulative Test for Chapters P–2

Take this test to review the material from earlier chapters. When you are finished, check your work against the answers given in the back of the book.

In Exercises 1 and 2, simplify the expression.

1. $\dfrac{8x^2y^{-3}}{30x^{-1}y^2}$

2. $\sqrt{24x^4y^3}$

In Exercises 3–5, perform the operation and simplify the result.

3. $4x - [2x + 3(2 - x)]$

4. $(x - 2)(x^2 + x - 3)$

5. $\dfrac{2}{s + 3} - \dfrac{1}{s + 1}$

In Exercises 6–8, factor the expression completely.

6. $25 - (x - 2)^2$

7. $x - 5x^2 - 6x^3$

8. $54 - 16x^3$

In Exercises 9–11, graph the equation without using a graphing utility.

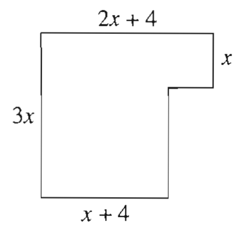
9. $x - 3y + 12 = 0$

10. $y = x^2 - 9$

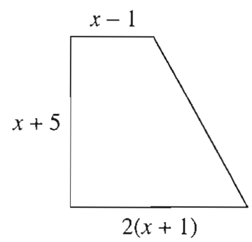
11. $y = \sqrt{4 - x}$

In Exercises 12 and 13, write an expression for the area of the region.

12.



13.



In Exercises 14–19, solve the equation by any convenient method. State the method you used.

14. $x^2 - 4x + 3 = 0$

15. $-2x^2 + 8x + 12 = 0$

16. $\frac{3}{4}x^2 = 12$

17. $3x^2 + 5x - 6 = 0$

18. $3x^2 + 9x + 1 = 0$

19. $\frac{1}{2}x^2 - 7 = 25$

In Exercises 20–25, solve the equation (if possible).

20. $x^4 + 12x^3 + 4x^2 + 48x = 0$

21. $8x^3 - 48x^2 + 72x = 0$

22. $x^{2/3} + 13 = 17$

23. $\sqrt{x + 10} = x - 2$

24. $|4(x - 2)| = 28$

25. $|x - 12| = -2$

In Exercises 26–28, determine whether each value of *x* is a solution of the inequality.

26. $4x + 2 > 7$

(a) $x = -1$ (b) $x = \frac{1}{2}$

(c) $x = \frac{3}{2}$ (d) $x = 2$

27. $3 - \frac{1}{2}x \le -2$

(a) $x = -10$ (b) $x = 9$

(c) $x = 10$ (d) $x = 12$

28. $|5x - 1| < 4$

(a) $x = -1$ (b) $x = -\frac{1}{2}$

(c) $x = 1$ (d) $x = 2$

In Exercises 29–32, solve the inequality and sketch the solution on the real number line.

29. $|x + 1| \le 6$

30. $|7 + 8x| > 5$

31. $5x^2 + 12x + 7 \ge 0$

32. $-x^2 + x + 4 < 0$

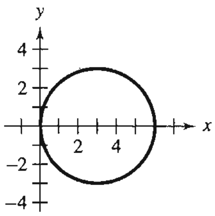


FIGURE FOR 34

33. Find an equation of the line passing through $\left(-\frac{1}{2}, 1\right)$ and $(3, 8)$.

34. Explain why the graph at the left does not represent y as a function of x.

35. Evaluate (if possible) the function $f(x) = \dfrac{x}{x - 2}$ for each value.

(a) $f(6)$ (b) $f(2)$ (c) $f(s + 2)$

36. Describe how the graph of each function would differ from the graph of $y = \sqrt[3]{x}$. (*Note:* It is not necessary to sketch the graphs.)

(a) $r(x) = \frac{1}{2}\sqrt[3]{x}$ (b) $h(x) = \sqrt[3]{x} + 2$ (c) $g(x) = \sqrt[3]{x + 2}$

In Exercises 37 and 38, find (a) $(f + g)(x)$, (b) $(f - g)(x)$, (c) $(fg)(x)$, and (d) $(f/g)(x)$. What is the domain of f/g?

37. $f(x) = x - 3,\quad g(x) = 4x + 1$

38. $f(x) = \sqrt{x - 1},\quad g(x) = x^2 + 1$

In Exercises 39 and 40, find (a) $f \circ g$ and (b) $g \circ f$. Find the domain of each composite function.

39. $f(x) = 2x^2,\quad g(x) = \sqrt{x + 6}$

40. $f(x) = x - 2,\quad g(x) = |x|$

41. Determine whether $h(x) = 5x - 2$ has an inverse function. If so, find it.

42. A group of n people decide to buy a \$36,000 minibus. Each person will pay an equal share of the cost. If three additional people join the group, the cost per person will decrease by \$1000. Find n.

43. For groups of 80 or more, a charter bus company determines the rate per person according to the formula

$$\text{Rate} = \$8.00 - \$0.05(n - 80), \quad n \ge 80.$$

(a) Write the revenue R as a function of n.

(b) Use a graphing utility to graph the revenue function. Move the cursor along the function to estimate the number of passengers that will maximize the revenue.

Proofs in Mathematics

Biconditional Statements

Recall from the Proofs in Mathematics in Chapter 1 that a conditional statement is a statement of the form "if p, then q." A statement of the form "p if and only if q" is called a **biconditional statement.** A biconditional statement, denoted by

$p \leftrightarrow q$ Biconditional statement

is the conjunction of the conditional statement $p \rightarrow q$ and its converse $q \rightarrow p$.

A biconditional statement can be either true or false. To be true, *both* the conditional statement and its converse must be true.

Example 1 Analyzing a Biconditional Statement

Consider the statement $x = 3$ if and only if $x^2 = 9$.

a. Is the statement a biconditional statement? **b.** Is the statement true?

Solution

a. The statement is a biconditional statement because it is of the form "p if and only if q."

b. The statement can be rewritten as the following conditional statement and its converse.

Conditional statement: If $x = 3$, then $x^2 = 9$.
Converse: If $x^2 = 9$, then $x = 3$.

The first of these statements is true, but the second is false because x could also equal -3. So, the biconditional statement is false.

Knowing how to use biconditional statements is an important tool for reasoning in mathematics.

Example 2 Analyzing a Biconditional Statement

Determine whether the biconditional statement is true or false. If it is false, provide a counterexample.

A number is divisible by 5 if and only if it ends in 0.

Solution

The biconditional statement can be rewritten as the following conditional statement and its converse.

Conditional statement: If a number is divisible by 5, then it ends in 0.
Converse: If a number ends in 0, then it is divisible by 5.

The conditional statement is false. A counterexample is the number 15, which is divisible by 5 but does not end in 0.

P.S. Problem Solving

1. As a salesperson, you receive a monthly salary of \$2000, plus a commission of 7% of sales. You are offered a new job at \$2300 per month, plus a commission of 5% of sales.
   (a) Write a linear equation for your current monthly wage W_1 in terms of your monthly sales S.
   (b) Write a linear equation for the monthly wage W_2 of your new job offer in terms of the monthly sales S.
   (c) Use a graphing utility to graph both equations in the same viewing window. Find the point of intersection. What does it signify?
   (d) You think you can sell \$20,000 per month. Should you change jobs? Explain.
2. For the numbers 2 through 9 on a telephone keypad (see figure), create two relations: one mapping numbers onto letters, and the other mapping letters onto numbers. Are both relations functions? Explain.

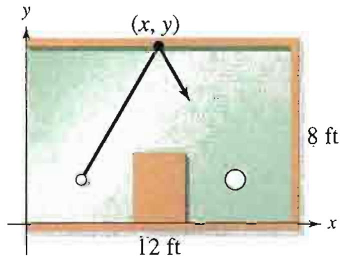
3. What can be said about the sum and difference of each of the following?
   (a) Two even functions
   (b) Two odd functions
   (c) An odd function and an even function
4. The two functions

   $f(x) = x$ and $g(x) = -x$

   are their own inverse functions. Graph each function and explain why this is true. Graph other linear functions that are their own inverse functions. Find a general formula for a family of linear functions that are their own inverse functions.
5. Prove that a function of the following form is even.

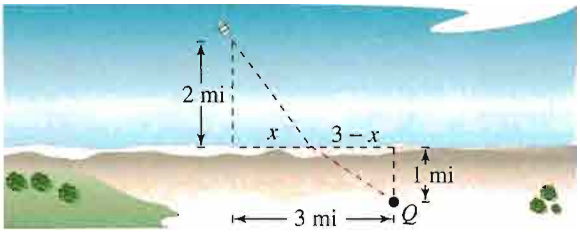
   $y = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdot \cdot \cdot + a_2x^2 + a_0$

6. A miniature golf professional is trying to make a hole-in-one on the miniature golf green shown. A coordinate plane is placed over the golf green. The golf ball is at the point (2.5, 2) and the hole is at the point (9.5, 2). The professional wants to bank the ball off the side wall of the green at the point (x, y). Find the coordinates of the point (x, y). Then write an equation for the path of the ball.



7. At 2:00 P.M. on April 11, 1912, the *Titanic* left Cobh, Ireland, on her voyage to New York City. At 11:40 P.M. on April 14, the *Titanic* struck an iceberg and sank, having covered only about 2100 miles of the approximately 3400-mile trip.
   (a) What was the total length of the *Titantic's* voyage in hours?
   (b) What was the *Titantic's* average speed in miles per hour?
   (c) Write a function relating the *Titantic's* distance from New York City and the number of hours traveled. Find the domain and range of the function.
   (d) Graph the function from part (c).
8. Consider the functions $f(x) = 4x$ and $g(x) = x + 6$.
   (a) Find $(f \circ g)(x)$.
   (b) Find $(f \circ g)^{-1}(x)$.
   (c) Find $f^{-1}(x)$ and $g^{-1}(x)$.
   (d) Find $(g^{-1} \circ f^{-1})(x)$ and compare the result with that of part (b).
   (e) Repeat parts (a) through (d) for $f(x) = x^3 + 1$ and $g(x) = 2x$.
   (f) Write two one-to-one functions f and g, and repeat parts (a) through (d) for these functions.
   (g) Make a conjecture about $(f \circ g)^{-1}(x)$ and $(g^{-1} \circ f^{-1})(x)$.

9. You are in a boat 2 miles from the nearest point on the coast. You are to travel to a point Q, 3 miles down the coast and 1 mile inland (see figure). You can row at 2 miles per hour and walk at 4 miles per hour.



(a) Write the total time T of the trip as a function of x.

(b) Determine the domain of the function.

(c) Use a graphing utility to graph the function. Be sure to choose an appropriate viewing window.

(d) Use the *zoom* and *trace* features to find the value of x that minimizes T.

(e) Write a brief paragraph interpreting these values.

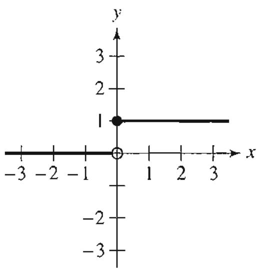
10. The Heaviside function $H(x)$ is widely used in engineering applications. (See figure.) To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

$$H(x) = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

Sketch the graph of each function by hand.

(a) $H(x) - 2$ (b) $H(x - 2)$ (c) $-H(x)$

(d) $H(-x)$ (e) $\frac{1}{2}H(x)$ (f) $-H(x - 2) + 2$



11. Let $f(x) = \dfrac{1}{1 - x}$.

(a) What are the domain and range of f?

(b) Find $f(f(x))$. What is the domain of this function?

(c) Find $f(f(f(x)))$. Is the graph a line? Why or why not?

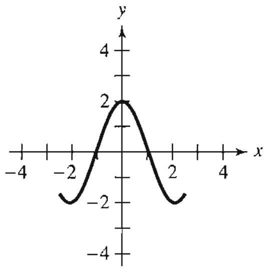
12. Show that the Associative Property holds for compositions of functions—that is,

$$(f \circ (g \circ h))(x) = ((f \circ g) \circ h)(x).$$

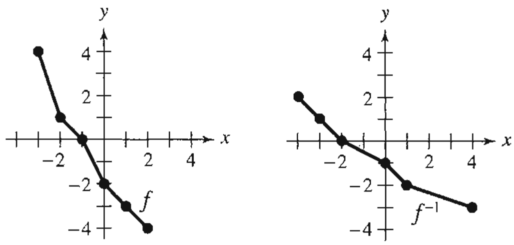
13. Consider the graph of the function f shown in the figure. Use this graph to sketch the graph of each function. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

(a) $f(x + 1)$ (b) $f(x) + 1$ (c) $2f(x)$ (d) $f(-x)$

(e) $-f(x)$ (f) $|f(x)|$ (g) $f(|x|)$



14. Use the graphs of f and f^{-1} to complete each table of function values.



(a)

x	$f(f^{-1}(x))$
-4	
-2	
0	
4	

(b)

x	$(f + f^{-1})(x)$
-3	
-2	
0	
1	

(c)

x	$(f \cdot f^{-1})(x)$
-3	
-2	
0	
1	

(d)

x	$\lvert f^{-1}(x)\rvert$
-4	
-3	
0	
4	