How to study Chapter 3

What you should learn

In this chapter you will learn the following skills and concepts:

- How to sketch and analyze graphs of functions
- How to sketch and analyze graphs of polynomial functions
- How to use long division and synthetic division to divide polynomials by other polynomials
- How to determine the number of rational and real zeros of polynomial functions, and find the zeros
- How to write mathematical models for direct, inverse, and joint variation

Important Vocabulary

As you encounter each new vocabulary term in this chapter, add the term and its definition to your notebook glossary.

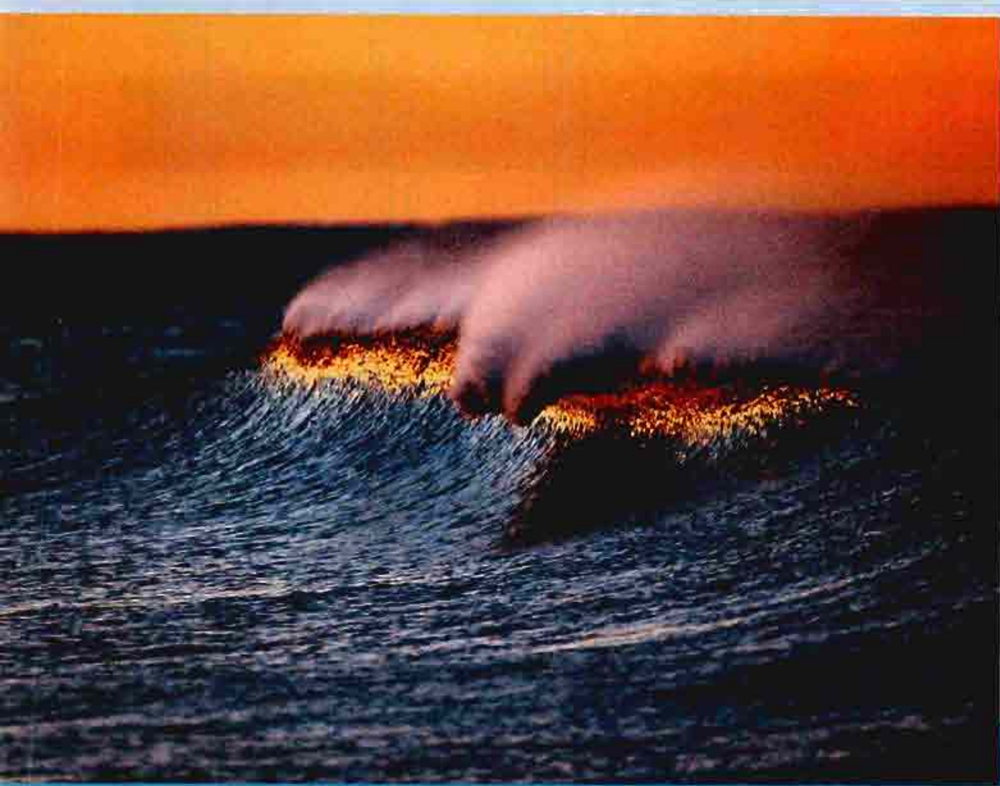
Polynomial function (p. 260)
Parabola (p. 260)
Axis (of a parabola) (p. 261)
Vertex (of a parabola) (p. 261)
Standard form of a quadratic function (p. 263)
Continuous (p. 271)
Leading Coefficient Test (p. 273)
Repeated zero (p. 275)
Multiplicity (p. 275)
Intermediate Value Theorem (p. 278)
Division Algorithm (p. 285)
Improper (rational expression) (p. 285)
Proper (rational expression) (p. 285)
Synthetic division (p. 287)
Rational Zero Test (p. 294)
Conjugates (p. 297)
Irreducible over the reals (p. 298)
Descartes's Rule of Signs (p. 300)
Variation in sign (p. 300)
Upper bound (p. 301)
Lower bound (p. 301)
Directly proportional (p. 309)
Constant of variation (p. 309)
Inversely proportional (p. 311)
Jointly proportional (p. 312)
Sum of square differences (p. 313)
Least squares regression line (p. 313)

Study Tools

Learning objectives in each section
Chapter Summary (p. 320)
Review Exercises (pp. 321–324)
Chapter Test (p. 325)

Additional Resources

Study and Solutions Guide
Interactive College Algebra
Videotapes/DVD for Chapter 3
College Algebra Website
Student Success Organizer



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3

Polynomial Functions

3.1 Quadratic Functions

▶ What you should learn

- How to analyze graphs of quadratic functions
- How to write quadratic functions in standard form and use the results to sketch graphs of functions
- How to use quadratic functions to model and solve real-life problems

▶ Why you should learn it

Quadratic functions can be used to model data to analyze consumer behavior. For instance, in Exercise 86 on page 269, you will use a quadratic function to model the number of hairdressers and cosmetologists in the United States.

Jeff Greenberg/The Image Works

The Graph of a Quadratic Function

In this and the next section you will study the graphs of polynomial functions. In Section 2.4, you were introduced to the following basic functions.

$f(x) = ax + b$ Linear function

$f(x) = c$ Constant function

$f(x) = x^2$ Squaring function

These functions are examples of **polynomial functions.**

Definition of Polynomial Function

Let n be a nonnegative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. The function

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

is called a **polynomial function of x with degree n.**

Polynomial functions are classified by degree. For instance, a constant function has degree 0 and a linear function has degree 1. In this section you will study second-degree polynomial functions, which are called **quadratic functions.**

For instance, each of the following functions is a quadratic function.

$$f(x) = x^2 + 6x + 2$$

$$g(x) = 2(x + 1)^2 - 3$$

$$h(x) = 9 + \tfrac{1}{4}x^2$$

$$k(x) = -3x^2 + 4$$

$$m(x) = (x - 2)(x + 1)$$

Note that the squaring function is a simple quadratic function that has degree 2.

Definition of Quadratic Function

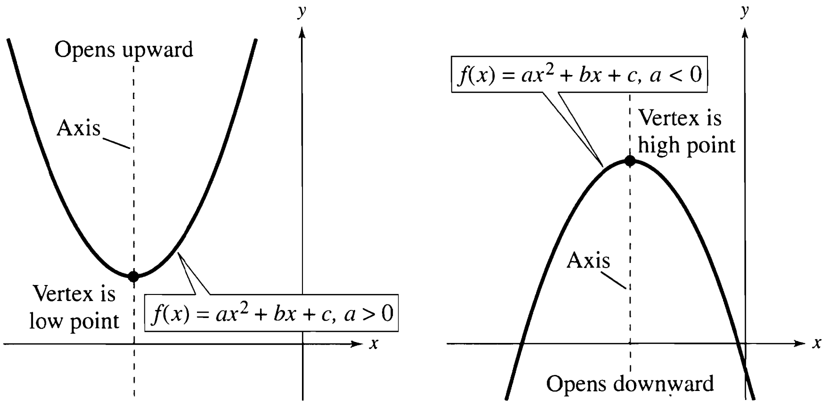
Let a, b, and c be real numbers with $a \neq 0$. The function

$f(x) = ax^2 + bx + c$ Quadratic function

is called a **quadratic function.**

The graph of a quadratic function is a special type of "U"-shaped curve called a **parabola.** Parabolas occur in many real-life applications—especially those involving reflective properties of satellite dishes and flashlight reflectors. You will study these properties in Section 4.4.

All parabolas are symmetric with respect to a line called the **axis of symmetry,** or simply the **axis** of the parabola. The point where the axis intersects the parabola is the **vertex** of the parabola, as shown in Figure 3.1. If the leading coefficient is positive, the graph of $f(x) = ax^2 + bx + c$ is a parabola that opens upward. If the leading coefficient is negative, the graph of $f(x) = ax^2 + bx + c$ is a parabola that opens downward.



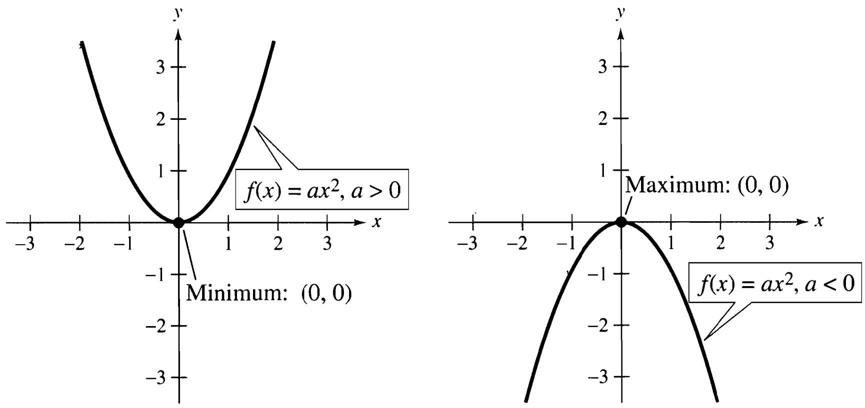
Leading coefficient is positive. *Leading coefficient is negative.*

FIGURE 3.1

The simplest type of quadratic function is

$$f(x) = ax^2.$$

Its graph is a parabola whose vertex is (0, 0). If $a > 0$, the vertex is the point with the *minimum* y-value on the graph, and if $a < 0$, the vertex is the point with the *maximum* y-value on the graph, as shown in Figure 3.2.



Leading coefficient is positive. *Leading coefficient is negative.*

FIGURE 3.2

When sketching the graph of $f(x) = ax^2$, it is helpful to use the graph of $y = x^2$ as a reference, as discussed in Section 2.5.

Exploration

Graph $y = ax^2$ for $a = -2, -1, -0.5, 0.5, 1$, and 2. How does changing the value of a affect the graph?

Graph $y = (x - h)^2$ for $h = -4, -2, 2$, and 4. How does changing the value of h affect the graph?

Graph $y = x^2 + k$ for $k = -4, -2, 2$, and 4. How does changing the value of k affect the graph?

The *Interactive* CD-ROM and *Internet* versions of this text offer a Try It for each example in the text.

Example 1 ▶ Sketching Graphs of Quadratic Functions

a. Compare the graphs of $y = x^2$ and $f(x) = \frac{1}{3}x^2$.

b. Compare the graphs of $y = x^2$ and $g(x) = 2x^2$.

Solution

a. Compared with $y = x^2$, each output of $f(x) = \frac{1}{3}x^2$ "shrinks" by a factor of $\frac{1}{3}$, creating the broader parabola shown in Figure 3.3.

b. Compared with $y = x^2$, each output of $g(x) = 2x^2$ "stretches" by a factor of 2, creating the narrower parabola shown in Figure 3.4.

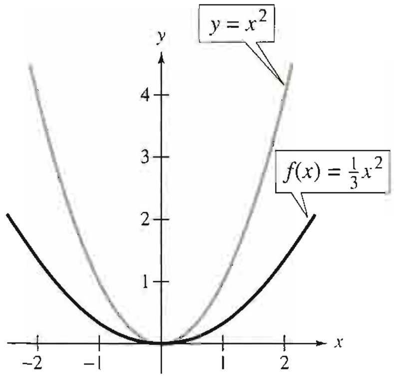


FIGURE 3.3

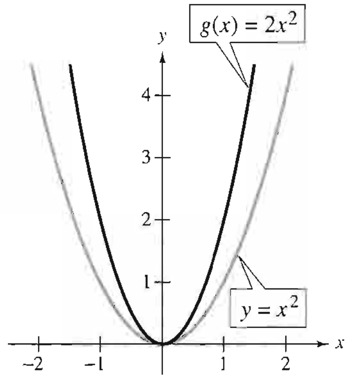


FIGURE 3.4

In Example 1, note that the coefficient a determines how widely the parabola given by $f(x) = ax^2$ opens. If $|a|$ is small, the parabola opens more widely than if $|a|$ is large.

Recall from Section 2.5 that the graphs of $y = f(x \pm c)$, $y = f(x) \pm c$, $y = f(-x)$, and $y = -f(x)$ are rigid transformations of the graph of $y = f(x)$. For instance, in Figure 3.5, notice how the graph of $y = x^2$ can be transformed to produce the graphs of $f(x) = -x^2 + 1$ and $g(x) = (x + 2)^2 - 3$.

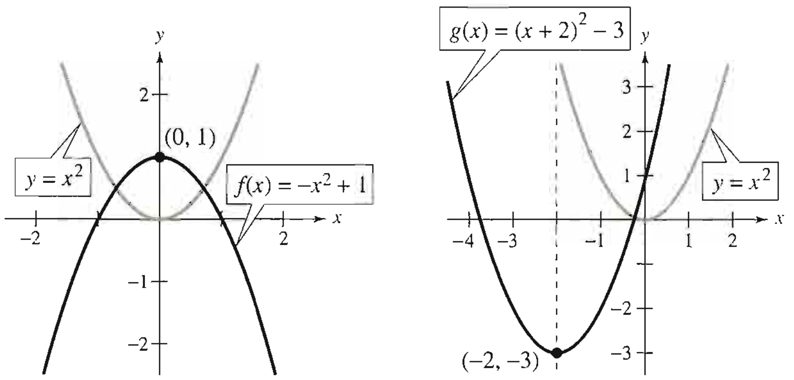


FIGURE 3.5

STUDY TIP

The standard form of a quadratic function identifies four basic transformations of the graph of $y = x^2$.

a. The factor $|a|$ produces a vertical stretch or shrink.

b. If $a < 0$, the graph is reflected in the x-axis.

c. The factor $(x - h)^2$ represents a horizontal shift of h units.

d. The term k represents a vertical shift of k units.

The Standard Form of a Quadratic Function

The **standard form** of a quadratic function is

$$f(x) = a(x - h)^2 + k.$$

This form is especially convenient for sketching a parabola because it identifies the vertex of the parabola.

Standard Form of a Quadratic Function

The quadratic function

$$f(x) = a(x - h)^2 + k, \qquad a \neq 0$$

is in **standard form.** The graph of f is a parabola whose axis is the vertical line $x = h$ and whose vertex is the point (h, k). If $a > 0$, the parabola opens upward, and if $a < 0$, the parabola opens downward.

To graph a parabola, it is helpful to begin by writing the quadratic function in standard form using the process of completing the square, as illustrated in Example 2.

Example 2 ▶ Graphing a Parabola in Standard Form

Sketch the graph of

$$f(x) = 2x^2 + 8x + 7$$

and identify the vertex and the axis of the parabola.

Solution

Begin by writing the quadratic function in standard form. Notice that the first step in completing the square is to factor out any coefficient of x^2 that is not 1.

$$\begin{aligned} f(x) &= 2x^2 + 8x + 7 && \text{Write original function.}\\ &= 2(x^2 + 4x) + 7 && \text{Factor 2 out of } x\text{-terms.}\\ &= 2(x^2 + 4x + \underbrace{4}_{(4/2)^2} - 4) + 7 && \text{Add and subtract 4 within parentheses.}\\ &= 2(x^2 + 4x + 4) - 2(4) + 7 && \text{Regroup terms.}\\ &= 2(x^2 + 4x + 4) - 8 + 7 && \text{Simplify.}\\ &= 2(x + 2)^2 - 1 && \text{Write in standard form.} \end{aligned}$$

From this form, you can see that the graph of f is a parabola that opens upward and has its vertex at $(-2, -1)$. This corresponds to a left shift of two units and a downward shift of one unit relative to the graph of $y = 2x^2$, as shown in Figure 3.6. In the figure, you can see that the axis of the parabola is the vertical line through the vertex, $x = -2$.

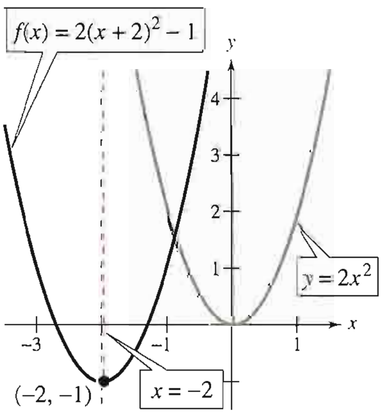


FIGURE 3.6

The icon identifies examples and concepts related to features of the Learning Tools CD-ROM and the *Interactive* and *Internet* versions of this text. For more details see the chart on pages *xix–xxiii*.

The *Interactive* CD-ROM and *Internet* versions of this text offer a Quiz for every section of the text.

To find the x-intercepts of the graph of $f(x) = ax^2 + bx + c$, you must solve the equation $ax^2 + bx + c = 0$. If $ax^2 + bx + c$ does not factor, you can use the Quadratic Formula to find the x-intercepts. Remember, however, that a parabola may have no x-intercepts.

Example 3 ▶ Finding the Vertex and x-Intercepts of a Parabola

Sketch the graph of $f(x) = -x^2 + 6x - 8$ and identify the vertex and x-intercepts.

Solution

As in Example 2, begin by writing the quadratic function in standard form.

$$f(x) = -x^2 + 6x - 8 \qquad \text{Write original function.}$$

$$= -(x^2 - 6x) - 8 \qquad \text{Factor } -1 \text{ out of } x\text{-terms.}$$

$$= -(x^2 - 6x + \underbrace{9}_{(-6/2)^2} - 9) - 8 \qquad \text{Add and subtract 9 within parentheses.}$$

$$= -(x^2 - 6x + 9) - (-9) - 8 \qquad \text{Regroup terms.}$$

$$= -(x - 3)^2 + 1 \qquad \text{Write in standard form.}$$

From this form, you can see that the vertex is $(3, 1)$. To find the x-intercepts of the graph, solve the equation $-x^2 + 6x - 8 = 0$.

$$-(x^2 - 6x + 8) = 0 \qquad \text{Factor out } -1.$$

$$-(x - 2)(x - 4) = 0 \qquad \text{Factor.}$$

$$x - 2 = 0 \quad \Rightarrow \quad x = 2 \qquad \text{Set 1st factor equal to 0.}$$

$$x - 4 = 0 \quad \Rightarrow \quad x = 4 \qquad \text{Set 2nd factor equal to 0.}$$

The x-intercepts are $(2, 0)$ and $(4, 0)$. So, the graph of f is a parabola that opens downward, as shown in Figure 3.7.

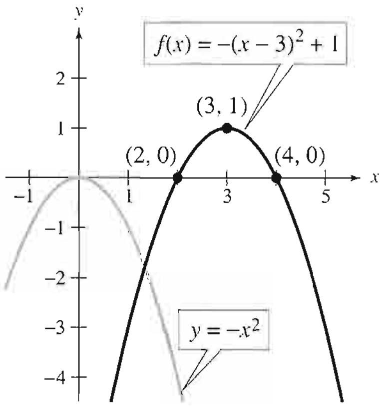


FIGURE 3.7

Example 4 ▶ Writing the Equation of a Parabola

Write the standard form of the equation of the parabola whose vertex is $(1, 2)$ and that passes through the point $(0, 0)$, as shown in Figure 3.8.

Solution

Because the vertex of the parabola is at $(h, k) = (1, 2)$, the equation has the form

$$f(x) = a(x - 1)^2 + 2. \qquad \text{Substitute for } h \text{ and } k \text{ in standard form.}$$

Because the parabola passes through the point $(0, 0)$, it follows that $f(0) = 0$. So,

$$0 = a(0 - 1)^2 + 2 \quad \Rightarrow \quad a = -2 \qquad \text{Substitute 0 for } x\text{; solve for } a.$$

which implies that the equation is

$$f(x) = -2(x - 1)^2 + 2. \qquad \text{Substitute for } a \text{ in standard form.}$$

So, the equation of this parabola is $y = -2(x - 1)^2 + 2$.

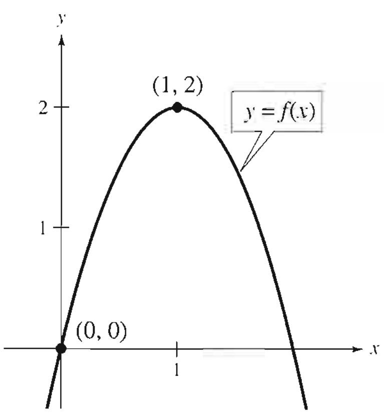


FIGURE 3.8

Application

Many applications involve finding the maximum or minimum value of a quadratic function. Some quadratic functions are not easily written in standard form. For such functions, it is useful to have an alternative method for finding the vertex. For a quadratic function in the form $f(x) = ax^2 + bx + c$, the vertex occurs when $x = -b/2a$. (You are asked to verify this in Exercise 91.)

Vertex of a Parabola

The vertex of the graph of $f(x) = ax^2 + bx + c$ is $\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$.

Example 5 ▶ The Maximum Height of a Baseball

A baseball is hit at a point 3 feet above the ground at a velocity of 100 feet per second and at an angle of 45° with respect to the ground. The path of the baseball is given by the function

$$f(x) = -0.0032x^2 + x + 3$$

where $f(x)$ is the height of the baseball (in feet) and x is the horizontal distance from home plate (in feet). What is the maximum height reached by the baseball?

Solution

For this quadratic function, you have

$$f(x) = ax^2 + bx + c$$
$$= -0.0032x^2 + x + 3.$$

So, $a = -0.0032$ and $b = 1$. Because the function has a maximum at $x = -b/2a$, you can conclude that the baseball reaches its maximum height when it is x feet from home plate, where x is

$$x = -\frac{b}{2a}$$

$$= -\frac{1}{2(-0.0032)} \qquad \text{Substitute for } a \text{ and } b.$$

$$= 156.25 \text{ feet.}$$

To find the maximum height, you must determine the value of the function when $x = 156.25$.

$$f(156.25) = -0.0032(156.25)^2 + 156.25 + 3$$
$$= 81.125 \text{ feet.}$$

The path of the baseball is shown in Figure 3.9. You can estimate from the graph in Figure 3.9 that the ball hits the ground at a distance of about 320 feet from home plate. The actual distance is the x-intercept of the graph of f, which you can find by solving the equation $-0.0032x^2 + x + 3 = 0$ and taking the positive solution, $x \approx 315.5$.

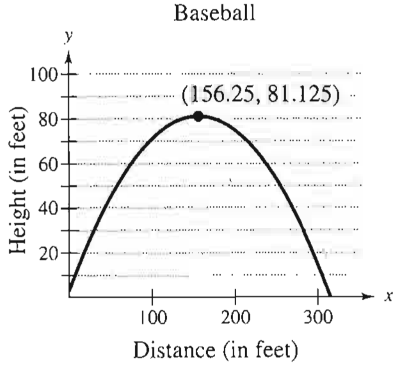


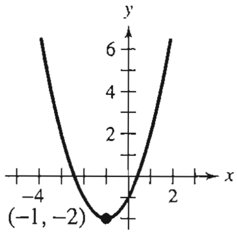
FIGURE 3.9

3.1 Exercises

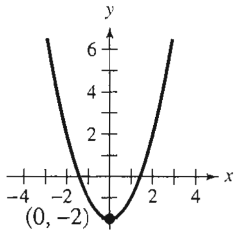
The *Interactive* CD-ROM and *Internet* versions of this text contain step-by-step solutions to all odd-numbered exercises. They also provide Tutorial Exercises for additional help.

In Exercises 1–8, match the quadratic function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

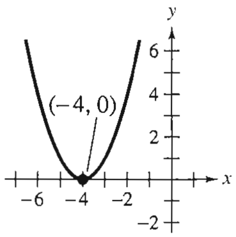
(a)



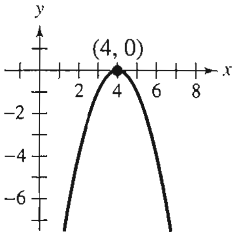
(b)



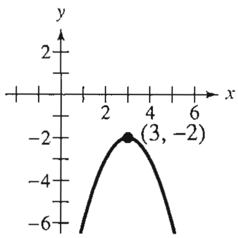
(c)



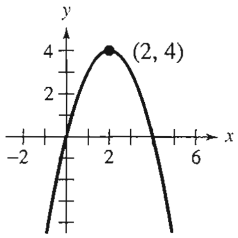
(d)



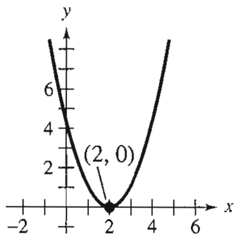
(e)



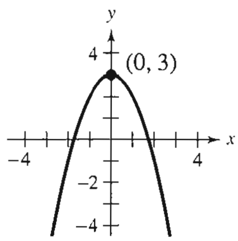
(f)



(g)



(h)



1. $f(x) = (x - 2)^2$

2. $f(x) = (x + 4)^2$

3. $f(x) = x^2 - 2$

4. $f(x) = 3 - x^2$

5. $f(x) = 4 - (x - 2)^2$

6. $f(x) = (x + 1)^2 - 2$

7. $f(x) = -(x - 3)^2 - 2$

8. $f(x) = -(x - 4)^2$

In Exercises 9–12, graph each function. Compare the graph of each function with the graph of $y = x^2$.

9. (a) $f(x) = \frac{1}{2}x^2$ (b) $g(x) = -\frac{1}{8}x^2$
(c) $h(x) = \frac{3}{2}x^2$ (d) $k(x) = -3x^2$

10. (a) $f(x) = x^2 + 1$ (b) $g(x) = x^2 - 1$
(c) $h(x) = x^2 + 3$ (d) $k(x) = x^2 - 3$

11. (a) $f(x) = (x - 1)^2$ (b) $g(x) = (3x)^2 + 1$
(c) $h(x) = \left(\frac{1}{3}x\right)^2 - 3$ (d) $k(x) = (x + 3)^2$

12. (a) $f(x) = -\frac{1}{2}(x - 2)^2 + 1$
(b) $g(x) = \left[\frac{1}{2}(x - 1)\right]^2 - 3$
(c) $h(x) = -\frac{1}{2}(x + 2)^2 - 1$
(d) $k(x) = [2(x + 1)]^2 + 4$

In Exercises 13–28, sketch the graph of the quadratic function without using a graphing utility. Identify the vertex and x-intercept(s).

13. $f(x) = x^2 - 5$

14. $h(x) = 25 - x^2$

15. $f(x) = \frac{1}{2}x^2 - 4$

16. $f(x) = 16 - \frac{1}{4}x^2$

17. $f(x) = (x + 5)^2 - 6$

18. $f(x) = (x - 6)^2 + 3$

19. $h(x) = x^2 - 8x + 16$

20. $g(x) = x^2 + 2x + 1$

21. $f(x) = x^2 - x + \frac{5}{4}$

22. $f(x) = x^2 + 3x + \frac{1}{4}$

23. $f(x) = -x^2 + 2x + 5$

24. $f(x) = -x^2 - 4x + 1$

25. $h(x) = 4x^2 - 4x + 21$

26. $f(x) = 2x^2 - x + 1$

27. $f(x) = \frac{1}{4}x^2 - 2x - 12$

28. $f(x) = -\frac{1}{3}x^2 + 3x - 6$

In Exercises 29–36, use a graphing utility to graph the quadratic function. Identify the vertex and x-intercepts. Then check your results algebraically by writing the quadratic function in standard form.

29. $f(x) = -(x^2 + 2x - 3)$

30. $f(x) = -(x^2 + x - 30)$

31. $g(x) = x^2 + 8x + 11$

32. $f(x) = x^2 + 10x + 14$

33. $f(x) = 2x^2 - 16x + 31$

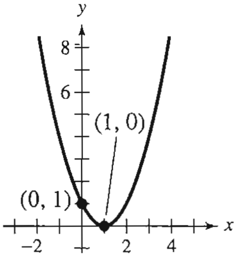
34. $f(x) = -4x^2 + 24x - 41$

35. $g(x) = \frac{1}{2}(x^2 + 4x - 2)$

36. $f(x) = \frac{3}{5}(x^2 + 6x - 5)$

In Exercises 37–42, find the standard form of the quadratic function.

37.



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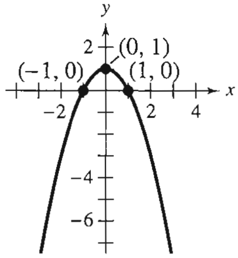

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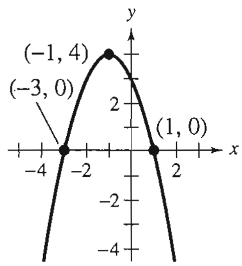

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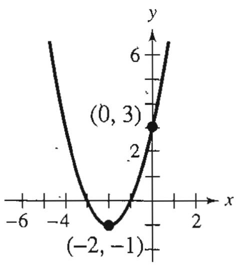

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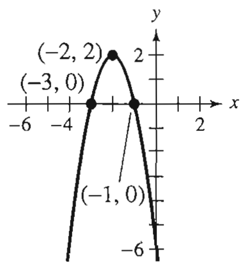

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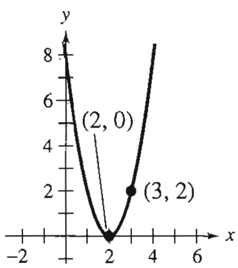


In Exercises 43–52, find the quadratic function that has the indicated vertex and whose graph passes through the given point.

43. Vertex: $(-2, 5)$; Point: $(0, 9)$

44. Vertex: $(4, -1)$; Point: $(2, 3)$

45. Vertex: $(3, 4)$; Point: $(1, 2)$

46. Vertex: $(2, 3)$; Point: $(0, 2)$

47. Vertex: $(5, 12)$; Point: $(7, 15)$

48. Vertex: $(-2, -2)$; Point: $(-1, 0)$

49. Vertex: $\left(-\frac{1}{4}, \frac{3}{2}\right)$; Point: $(-2, 0)$

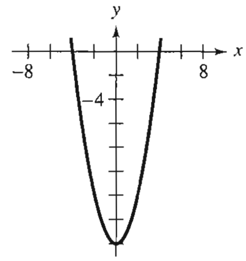
50. Vertex: $\left(\frac{5}{2}, -\frac{3}{4}\right)$; Point: $(-2, 4)$

51. Vertex: $\left(-\frac{5}{2}, 0\right)$; Point: $\left(-\frac{7}{2}, -\frac{16}{3}\right)$

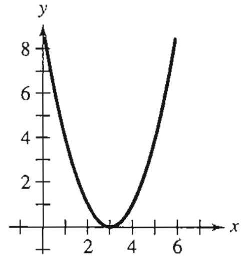
52. Vertex: $(6, 6)$; Point: $\left(\frac{61}{10}, \frac{3}{2}\right)$

Graphical Reasoning **In Exercises 53–56, determine the x-intercept(s) of the graph visually. Then find the x-intercepts algebraically to confirm your results.**

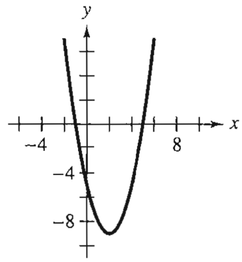
53. $y = x^2 - 16$



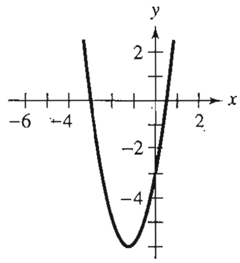
54. $y = x^2 - 6x + 9$



55. $y = x^2 - 4x - 5$



56. $y = 2x^2 + 5x - 3$



In Exercises 57–64, use a graphing utility to graph the quadratic function. Find the x-intercepts of the graph and compare them with the solutions of the corresponding quadratic equation when $y = 0$.

57. $f(x) = x^2 - 4x$

58. $f(x) = -2x^2 + 10x$

59. $f(x) = x^2 - 9x + 18$

60. $f(x) = x^2 - 8x - 20$

61. $f(x) = 2x^2 - 7x - 30$

62. $f(x) = 4x^2 + 25x - 21$

63. $f(x) = -\frac{1}{2}(x^2 - 6x - 7)$

64. $f(x) = \frac{7}{10}(x^2 + 12x - 45)$

In Exercises 65–70, find two quadratic functions, one that opens upward and one that opens downward, whose graphs have the given x-intercepts. (There are many correct answers.)

65. $(-1, 0), (3, 0)$

66. $(-5, 0), (5, 0)$

67. $(0, 0), (10, 0)$

68. $(4, 0), (8, 0)$

69. $(-3, 0), \left(-\frac{1}{2}, 0\right)$

70. $\left(-\frac{5}{2}, 0\right), (2, 0)$

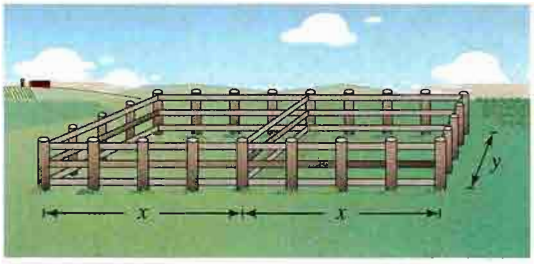
In Exercises 71–74, find two positive real numbers whose product is a maximum.

71. The sum is 110. **72.** The sum is S.

73. The sum of the first and twice the second is 24.

74. The sum of the first and three times the second is 42.

75. ***Numerical, Graphical, and Analytical Analysis*** A rancher has 200 feet of fencing to enclose two adjacent rectangular corrals (see figure).



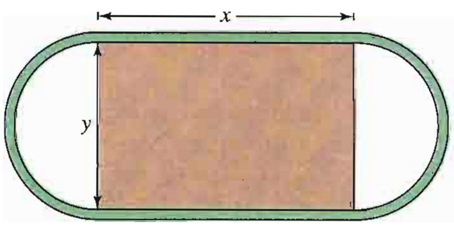
(a) Write the area A of the corral as a function of x.

(b) Create a table showing possible values of x and the corresponding area of the corral. Use the table to estimate the dimensions that will enclose the maximum area.

(c) Use a graphing utility to graph the area function. Use the graph to approximate the dimensions that will produce the maximum enclosed area.

(d) Write the area function in standard form to find analytically the dimensions that will produce the maximum area.

76. ***Geometry*** An indoor physical fitness room consists of a rectangular region with a semicircle on each end (see figure). The perimeter of the room is to be a 200-meter single-lane running track.



(a) Determine the radius of the semicircular ends of the room. Determine the distance, in terms of y, around the inside edge of the two semicircular parts of the track.

(b) Use the result of part (a) to write an equation, in terms of x and y, for the distance traveled in one lap around the track. Solve for y.

(c) Use the result of part (b) to write the area A of the rectangular region as a function of x. What dimensions will produce a maximum area of the rectangle?

77. ***Maximum Revenue*** Find the number of units sold that yields a maximum annual revenue for a company that produces health food supplements. The total revenue R (in dollars) is given by $R = 900x - 0.1x^2$, where x is the number of units sold.

78. ***Maximum Revenue*** Find the number of units sold that yields a maximum annual revenue for a sporting goods manufacturer. The total revenue R (in dollars) is given by $R = 100x - 0.0002x^2$, where x is the number of units sold.

79. ***Minimum Cost*** A manufacturer of lighting fixtures has daily production costs of

$$C = 800 - 10x + 0.25x^2$$

where C is the total cost (in dollars) and x is the number of units produced. How many fixtures should be produced each day to yield a minimum cost?

80. ***Minimum Cost*** A textile manufacturer has daily production costs of $C = 100{,}000 - 110x + 0.045x^2$, where C is the total cost (in dollars) and x is the number of units produced. How many units should be produced each day to yield a minimum cost?

81. ***Maximum Profit*** The profit P (in dollars) for a company that produces antivirus and system utilities software is

$$P = -0.0002x^2 + 140x - 250{,}000$$

where x is the number of units sold. What sales level will yield a maximum profit?

82. ***Maximum Profit*** The profit P (in hundreds of dollars) that a company makes depends on the amount x (in hundreds of dollars) the company spends on advertising according to the model

$$P = 230 + 20x - 0.5x^2.$$

What expenditure for advertising will yield a maximum profit?

83. ***Height of a Ball*** The height y (in feet) of a ball thrown by a child is

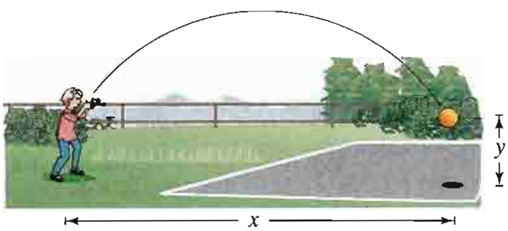
$$y = -\frac{1}{12}x^2 + 2x + 4$$

where x is the horizontal distance (in feet) from the point at which the ball is thrown (see figure).

(a) How high is the ball when it leaves the child's hand? (*Hint:* Find y when $x = 0$.)

(b) What is the maximum height of the ball?

(c) How far from the child does the ball strike the ground?



84. ***Path of a Diver*** The path of a diver is

$$y = -\frac{4}{9}x^2 + \frac{24}{9}x + 12$$

where y is the height (in feet) and x is the horizontal distance from the end of the diving board (in feet). What is the maximum height of the diver?

85. ***Graphical Analysis*** From 1960 to 2000, the per capita consumption C of cigarettes by Americans (age 18 and older) can be modeled by

$$C = 4258 + 6.5t - 1.62t^2, \qquad 0 \le t \le 40$$

where t is the year, with $t = 0$ corresponding to 1960. (Source: *Tobacco Situation and Outlook Yearbook*)

(a) Use a graphing utility to graph the model.

(b) Use the graph of the model to approximate the maximum average annual consumption. Beginning in 1966, all cigarette packages were required by law to carry a health warning. Do you think the warning had any effect? Explain.

(c) In 1990, the U.S. population (age 18 and over) was 185,105,441. Of those, about 63,423,167 were smokers. What was the average annual cigarette consumption *per smoker* in 1990? What was the average daily cigarette consumption *per smoker*?

▶ Model It

86. ***Data Analysis*** The numbers y (in thousands) of hairdressers and cosmetologists in the United States for the years 1995 through 2000 are shown in the table. (Source: U.S. Bureau of Labor Statistics)

Year	Number of hairdressers and cosmetologists, y
1995	750
1996	737
1997	748
1998	763
1999	784
2000	820

(a) Use a graphing utility to create a scatter plot of the data. Let x represent the year, with $x = 5$ corresponding to 1995.

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data.

(c) Use a graphing utility to graph the model in the same viewing window as the scatter plot. How well does the model fit the data?

(d) Use the *trace* feature of the graphing utility to approximate the year in which the number of hairdressers and cosmetologists was the least.

(e) Use the model to predict the number of hairdressers and cosmetologists in 2005.

87. ***Wind Drag*** The number of horsepower y required to overcome wind drag on an automobile is approximated by

$$y = 0.002s^2 + 0.005s - 0.029, \qquad 0 \le s \le 100$$

where s is the speed of the car (in miles per hour).

(a) Use a graphing utility to graph the function.

(b) Graphically estimate the maximum speed of the car if the power required to overcome wind drag is not to exceed 10 horsepower. Verify your estimate analytically.

88. ***Maximum Fuel Economy*** A study was done to compare the speed x (in miles per hour) with the mileage y (in miles per gallon) of an automobile. The results are shown in the table. (Source: Federal Highway Administration)

Speed, x	Mileage, y
15	22.3
20	25.5
25	27.5
30	29.0
35	28.8
40	30.0
45	29.9
50	30.2
55	30.4
60	28.8
65	27.4
70	25.3
75	23.3

(a) Use a graphing utility to create a scatter plot of the data.

(b) Use the *regression* feature of a graphing utility to find a quadratic model for the data.

(c) Use a graphing utility to graph the model in the same viewing window as the scatter plot.

(d) Estimate the speed for which the miles per gallon is greatest.

Synthesis

True or False? **In Exercises 89 and 90, determine whether the statement is true or false. Justify your answer.**

89. The function $f(x) = -12x^2 - 1$ has no x-intercepts.

90. The graphs of

$$f(x) = -4x^2 - 10x + 7$$

and

$$g(x) = 12x^2 + 30x + 1$$

have the same axis of symmetry.

91. Write the quadratic equation

$$f(x) = ax^2 + bx + c$$

in standard form to verify that the vertex occurs at

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right).$$

92. ***Profit*** The profit P (in millions of dollars) for a recreational vehicle retailer is modeled by a quadratic function of the form

$$P = at^2 + bt + c$$

where t represents the year. If you were president of the company, which of the models below would you prefer? Explain your reasoning.

(a) a is positive and $-b/(2a) \le t$.

(b) a is positive and $t \le -b/(2a)$.

(c) a is negative and $-b/(2a) \le t$.

(d) a is negative and $t \le -b/(2a)$.

93. Is it possible for a quadratic equation to have only one x-intercept? Explain.

94. Assume that the function

$$f(x) = ax^2 + bx + c, \qquad a \neq 0$$

has two real zeros. Show that the x-coordinate of the vertex of the graph is the average of the zeros of f. (*Hint:* Use the Quadratic Formula.)

Review

In Exercises 95–98, find the equation of the line in slope-intercept form that has the given characteristics.

95. Contains the points $(-4, 3)$ and $(2, 1)$

96. Contains the point $\left(\frac{7}{2}, 2\right)$ and has a slope of $\frac{3}{2}$

97. Contains the point $(0, 3)$ and is perpendicular to the line $4x + 5y = 10$

98. Contains the point $(-8, 4)$ and is parallel to the line $y = -3x + 2$

In Exercises 99–104, let $f(x) = 14x - 3$ and let $g(x) = 8x^2$. Find the indicated value.

99. $(f + g)(-3)$

100. $(g - f)(2)$

101. $(fg)\left(-\frac{4}{7}\right)$

102. $\left(\frac{f}{g}\right)(-1.5)$

103. $(f \circ g)(-1)$

104. $(g \circ f)(0)$

3.2 Polynomial Functions of Higher Degree

▶ What you should learn

- How to use transformations to sketch graphs of polynomial functions
- How to use the Leading Coefficient Test to determine the end behavior of graphs of polynomial functions
- How to use zeros of polynomial functions as sketching aids
- How to use the Intermediate Value Theorem to help locate zeros of polynomial functions

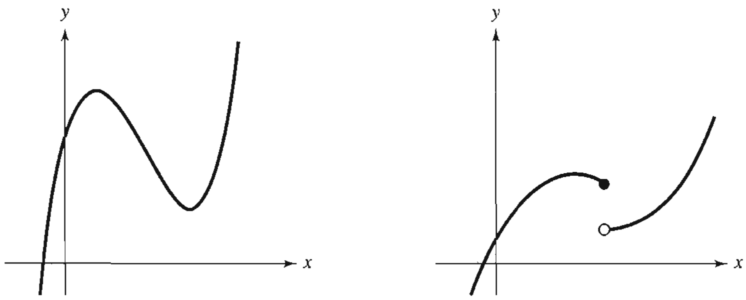
▶ Why you should learn it

You can use polynomial functions to model real-life processes, such as the growth of a red oak tree, as discussed in Exercise 91 on page 282.

Leonard Lee Rue/Earth Scenes

Graphs of Polynomial Functions

In this section you will study basic features of the graphs of polynomial functions. The first feature is that the graph of a polynomial function is **continuous.** Essentially, this means that the graph of a polynomial function has no breaks, holes, or gaps, as shown in Figure 3.10(a).

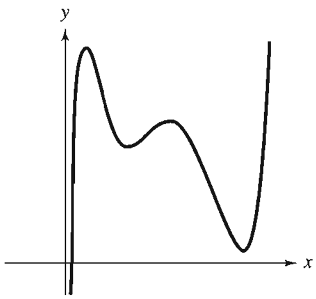


(a) Polynomial functions have continuous graphs.

(b) Functions with graphs that are not continuous are not polynomial functions.

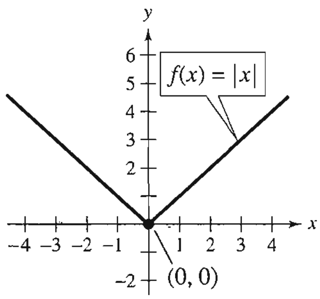
FIGURE 3.10

The second feature is that the graph of a polynomial function has only smooth, rounded turns, as shown in Figure 3.11. A polynomial function cannot have a sharp turn. For instance, the function $f(x) = |x|$, which has a sharp turn at the point $(0, 0)$, as shown in Figure 3.12, is not a polynomial function.



Polynomial functions have graphs with rounded turns.

FIGURE 3.11



Functions whose graphs have sharp turns are not polynomial functions.

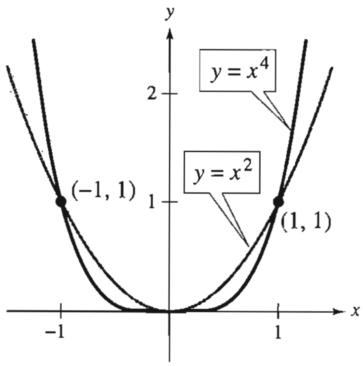
FIGURE 3.12

The graphs of polynomial functions of degree greater than 2 are more difficult to analyze than the graphs of polynomials of degree 0, 1, or 2. However, using the features presented in this section, together with point plotting, intercepts, and symmetry, you should be able to make reasonably accurate sketches *by hand.*

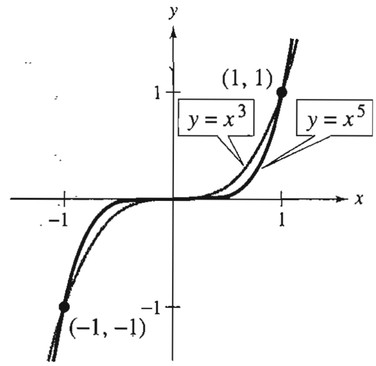
STUDY TIP

For power functions $f(x) = x^n$, if n is even, then the graph of the function is symmetric with respect to the y-axis, and if n is odd, then the graph of the function is symmetric with respect to the origin.

The polynomial functions that have the simplest graphs are monomials of the form $f(x) = x^n$, where n is an integer greater than zero. From Figure 3.13, you can see that when n is *even*, the graph is similar to the graph of $f(x) = x^2$, and when n is *odd*, the graph is similar to the graph of $f(x) = x^3$. Moreover, the greater the value of n, the flatter the graph near the origin. Polynomial functions of the form $f(x) = x^n$ are often referred to as **power functions.**



(a) If n is even, the graph of $y = x^n$ touches the axis at the x-intercept.



(b) If n is odd, the graph of $y = x^n$ crosses the axis at the x-intercept.

FIGURE 3.13

Example 1 ▶ Sketching Transformations of Monomial Functions

Sketch the graph of each function.

a. $f(x) = -x^5$

b. $h(x) = (x + 1)^4$

Solution

a. Because the degree of $f(x) = -x^5$ is odd, its graph is similar to the graph of $y = x^3$. In Figure 3.14, note that the negative coefficient has the effect of reflecting the graph in the x-axis.

b. The graph of $h(x) = (x + 1)^4$, as shown in Figure 3.15, is a left shift by one unit of the graph of $y = x^4$.

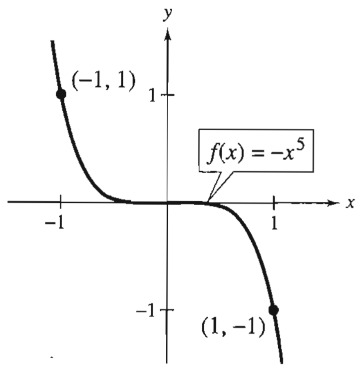


FIGURE 3.14

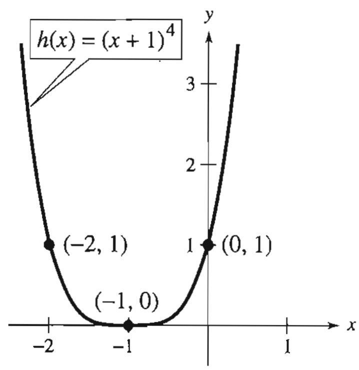


FIGURE 3.15

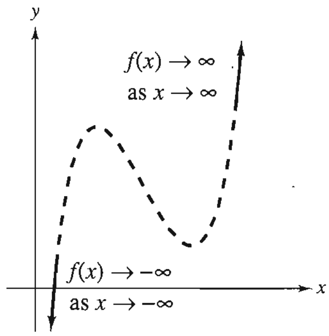
The Leading Coefficient Test

In Example 1, note that both graphs eventually rise or fall without bound as x moves to the right. Whether the graph of a polynomial function eventually rises or falls can be determined by the function's degree (even or odd) and by its leading coefficient, as indicated in the **Leading Coefficient Test.**

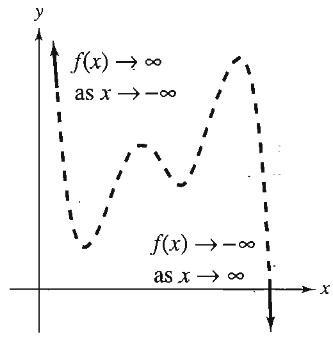
Leading Coefficient Test

As x moves without bound to the left or to the right, the graph of the polynomial function $f(x) = a_n x^n + \cdot \cdot \cdot + a_1 x + a_0$ eventually rises or falls in the following manner.

1. When n is *odd:*

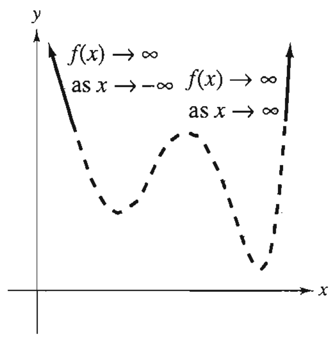


If the leading coefficient is positive ($a_n > 0$), the graph falls to the left and rises to the right.

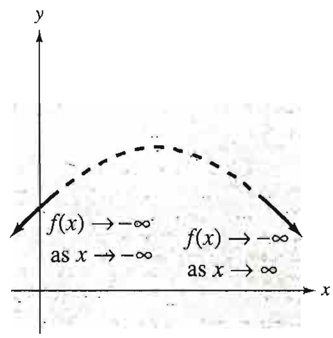


If the leading coefficient is negative ($a_n < 0$), the graph rises to the left and falls to the right.

2. When n is *even:*



If the leading coefficient is positive ($a_n > 0$), the graph rises to the left and right.



If the leading coefficient is negative ($a_n < 0$), the graph falls to the left and right.

The dashed portions of the graphs indicate that the test determines *only* the right-hand and left-hand behavior of the graph.

Exploration

For each function, identify the degree of the function and whether the degree of the function is even or odd. Identify the leading coefficient and whether the leading coefficient is greater than 0 or less than 0. Use a graphing utility to graph each function. Describe the relationship between the degree and the sign of the leading coefficient of the function and the right- and left-hand behavior of the graph of the function.

a. $f(x) = x^3 - 2x^2 - x + 1$

b. $f(x) = 2x^5 + 2x^2 - 5x + 1$

c. $f(x) = -2x^5 - x^2 + 5x + 3$

d. $f(x) = -x^3 + 5x - 2$

e. $f(x) = 2x^2 + 3x - 4$

f. $f(x) = x^4 - 3x^2 + 2x - 1$

g. $f(x) = x^2 + 3x + 2$

STUDY TIP

The notation "$f(x) \to -\infty$ as $x \to -\infty$" indicates that the graph falls to the left. The notation "$f(x) \to \infty$ as $x \to \infty$" indicates that the graph rises to the right.

Example 2 ▶ Applying the Leading Coefficient Test

Describe the right-hand and left-hand behavior of the graph of $f(x) = -x^3 + 4x$.

Solution

Because the degree is odd and the leading coefficient is negative, the graph rises to the left and falls to the right, as shown in Figure 3.16.

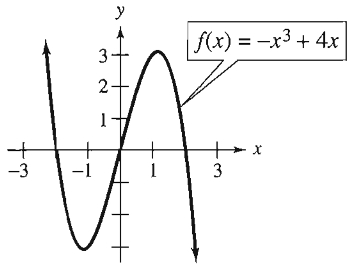


FIGURE 3.16

In Example 2, note that the Leading Coefficient Test tells you only whether the graph *eventually* rises or falls to the right or left. Other characteristics of the graph, such as intercepts and minimum and maximum points, must be determined by other tests.

Example 3 ▶ Applying the Leading Coefficient Test

Describe the right-hand and left-hand behavior of the graph of each function.

a. $f(x) = x^4 - 5x^2 + 4$ **b.** $f(x) = x^5 - x$

Solution

a. Because the degree is even and the leading coefficient is positive, the graph rises to the left and right, as shown in Figure 3.17.

b. Because the degree is odd and the leading coefficient is positive, the graph falls to the left and rises to the right, as shown in Figure 3.18.

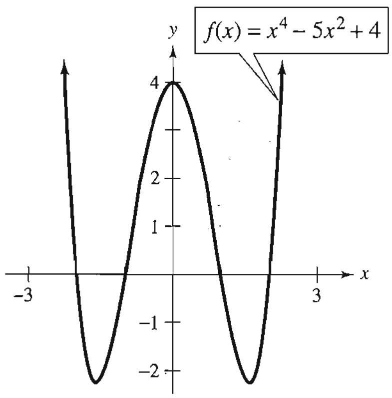


FIGURE 3.17

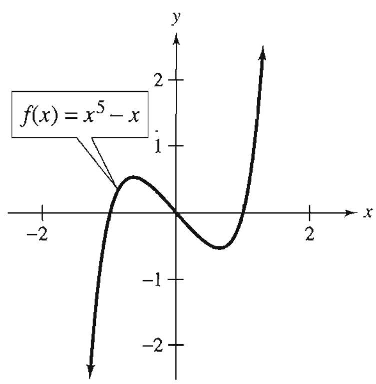


FIGURE 3.18

Zeros of Polynomial Functions

It can be shown that for a polynomial function f of degree n, the following statements are true.

1. The graph of f has, at most, $n - 1$ turning points. (Turning points are points at which the graph changes from increasing to decreasing or vice versa.)
2. The function f has, at most, n real zeros. (You will study this result in detail in Section 3.4 on the Fundamental Theorem of Algebra.)

Finding the zeros of polynomial functions is one of the most important problems in algebra. There is a strong interplay between graphical and algebraic approaches to this problem. Sometimes you can use information about the graph of a function to help find its zeros, and in other cases you can use information about the zeros of a function to help sketch its graph.

STUDY TIP

Remember that the *zeros* of a function of x are the x-values for which the function is zero.

Real Zeros of Polynomial Functions

If f is a polynomial function and a is a real number, the following statements are equivalent.

1. $x = a$ is a *zero* of the function f.
2. $x = a$ is a *solution* of the polynomial equation $f(x) = 0$.
3. $(x - a)$ is a *factor* of the polynomial $f(x)$.
4. $(a, 0)$ is an *x-intercept* of the graph of f.

Example 4 ▶ Finding the Zeros of a Polynomial Function

Find all real zeros of $f(x) = -2x^4 + 2x^2$. Use the graph in Figure 3.19 to determine the number of turning points of the graph of the function.

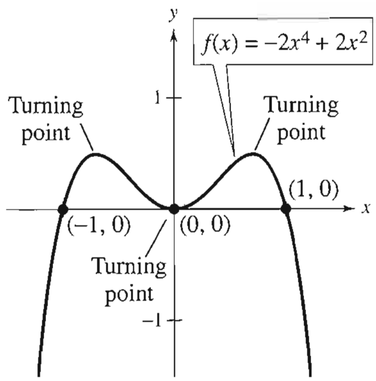


FIGURE 3.19

Solution

In this case, the polynomial factors as follows.

$$f(x) = -2x^2(x^2 - 1) \qquad \text{Remove common monomial factor.}$$

$$= -2x^2(x - 1)(x + 1) \qquad \text{Factor completely.}$$

So, the real zeros are $x = 0$, $x = 1$, and $x = -1$, and the corresponding x-intercepts are $(0, 0)$, $(1, 0)$, and $(-1, 0)$, as shown in Figure 3.19. Note in the figure that the graph has three turning points. This is consistent with the fact that a fourth-degree polynomial can have *at most* three turning points.

STUDY TIP

In Example 4, note that because k is even, the factor $-2x^2$ yields the repeated zero $x = 0$. The graph touches the x-axis at $x = 0$, as shown in Figure 3.19.

Repeated Zeros

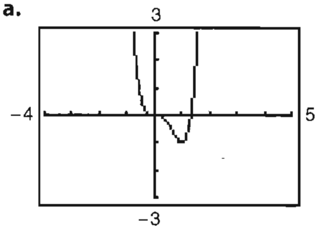
A factor $(x - a)^k$, $k > 1$, yields a **repeated zero** $x = a$ of **multiplicity** k.

1. If k is odd, the graph *crosses* the x-axis at $x = a$.
2. If k is even, the graph *touches* the x-axis (but does not cross the x-axis) at $x = a$.

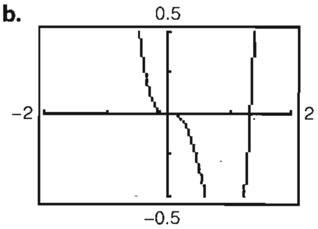
Technology

Example 5 uses an "algebraic approach" to describe the graph of the function. A graphing utility is a complement to this approach. Remember that an important aspect of using a graphing utility is to find a viewing window that shows all significant features of the graph. For instance, which of the graphs below shows all of the significant features of the function in Example 5?

a.



b.



Example 5 ▶ Sketching the Graph of a Polynomial Function

Sketch the graph of $f(x) = 3x^4 - 4x^3$.

Solution

1. *Apply the Leading Coefficient Test.* Because the leading coefficient is positive and the degree is even, you know that the graph eventually rises to the left and to the right (see Figure 3.20).
2. *Find the Zeros of the Polynomial.* By factoring

$$f(x) = 3x^4 - 4x^3 = x^3(3x - 4) \qquad \text{Remove common factor.}$$

you can see that the zeros of f are $x = 0$ and $x = \frac{4}{3}$ (both of odd multiplicity). So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{4}{3}, 0\right)$. Add these points to your graph, as shown in Figure 3.20.

3. *Plot a Few Additional Points.* To sketch the graph by hand, find a few additional points, as shown in the table. Then plot the points (see Figure 3.21).

x	$f(x)$
-1	7
0.5	-0.3125
1	-1
1.5	1.6875

4. *Draw the Graph.* Draw a continuous curve through the points, as shown in Figure 3.21. Because both zeros are of odd multiplicity, you know that the graph should cross the x-axis at $x = 0$ and $x = \frac{4}{3}$. If you are unsure of the shape of that portion of the graph, plot some additional points.

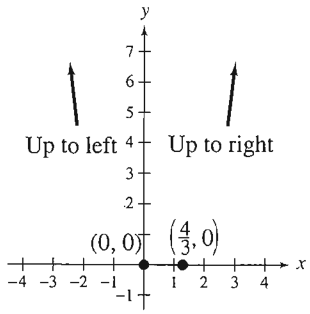


FIGURE 3.20

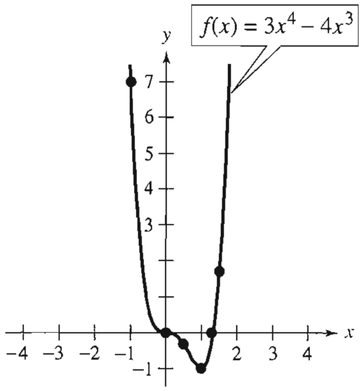


FIGURE 3.21

A polynomial function is written in **standard form** if its terms are written in descending order of exponents from left to right. Before applying the Leading Coefficient Test to a polynomial function, it is a good idea to check that the polynomial function is written in standard form.

Example 6 ▶ Sketching the Graph of a Polynomial Function

Sketch the graph of $f(x) = -2x^3 + 6x^2 - \frac{9}{2}x$.

Solution

1. *Apply the Leading Coefficient Test.* Because the leading coefficient is negative and the degree is odd, you know that the graph eventually rises to the left and falls to the right (see Figure 3.22).
2. *Find the Zeros of the Polynomial.* By factoring

$$f(x) = -2x^3 + 6x^2 - \frac{9}{2}x$$

$$= -\frac{1}{2}x(4x^2 - 12x + 9)$$

$$= -\frac{1}{2}x(2x - 3)^2$$

you can see that the zeros of f are $x = 0$ (odd multiplicity) and $x = \frac{3}{2}$ (even multiplicity). So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{3}{2}, 0\right)$. Add these points to your graph, as shown in Figure 3.22.

3. *Plot a Few Additional Points.* To sketch the graph by hand, find a few additional points, as shown in the table. Then plot the points (see Figure 3.23).

x	$f(x)$
-0.5	4
0.5	-1
1	-0.5
2	-1

4. *Draw the Graph.* Draw a continuous curve through the points, as shown in Figure 3.23. As indicated by the multiplicities of the zeros, the graph crosses the x-axis at $(0, 0)$ but does not cross the x-axis at $\left(\frac{3}{2}, 0\right)$.

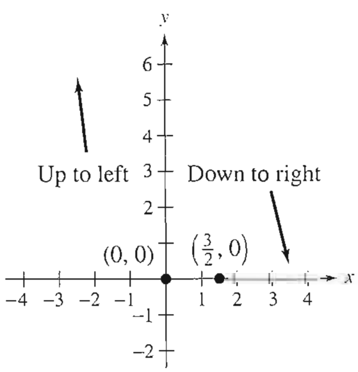


FIGURE 3.22

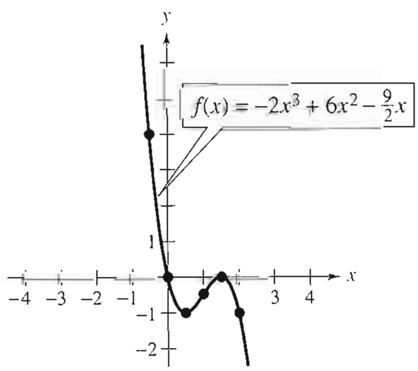


FIGURE 3.23

STUDY TIP

Observe in Example 6 that the sign of $f(x)$ is positive to the left of and negative to the right of the zero $x = 0$. Similarly, the sign of $f(x)$ is negative to the left and to the right of the zero $x = \frac{3}{2}$. This suggests that if the zero of a polynomial function is of *odd* multiplicity, then the sign of $f(x)$ changes from one side to the other side of the zero. If the zero is of *even* multiplicity, then the sign of $f(x)$ does not change from one side of the zero to the other side. The following table helps to illustrate this result.

x	-0.5	0	0.5
$f(x)$	4	0	-1
Sign	$+$		$-$

x	1	$\frac{3}{2}$	2
$f(x)$	-0.5	0	-1
Sign	$-$		$-$

This sign analysis may be helpful in graphing polynomial functions.

The Intermediate Value Theorem

The next theorem, called the **Intermediate Value Theorem,** tells you of the existence of real zeros of polynomial functions. This theorem implies that if $(a, f(a))$ and $(b, f(b))$ are two points on the graph of a polynomial function such that $f(a) \neq f(b)$, then for any number d between $f(a)$ and $f(b)$ there must be a number c between a and b such that $f(c) = d$. (See Figure 3.24.)

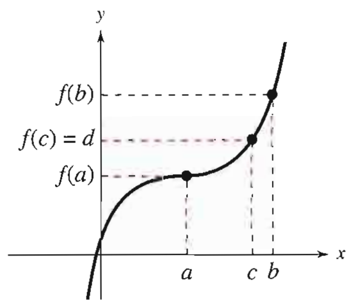


FIGURE 3.24

Intermediate Value Theorem

Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$, then, in the interval $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.

The Intermediate Value Theorem helps you locate the real zeros of a polynomial function in the following way. If you can find a value $x = a$ at which a polynomial function is positive, and another value $x = b$ at which it is negative, you can conclude that the function has at least one real zero between these two values. For example, the function $f(x) = x^3 + x^2 + 1$ is negative when $x = -2$ and positive when $x = -1$. Therefore, it follows from the Intermediate Value Theorem that f must have a real zero somewhere between -2 and -1, as shown in Figure 3.25.

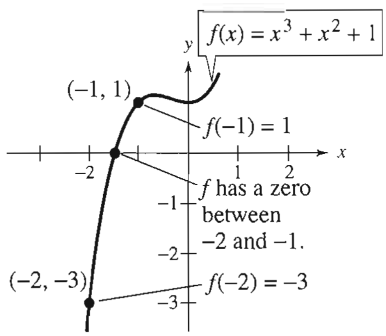


FIGURE 3.25

By continuing this line of reasoning, you can approximate any real zeros of a polynomial function to any desired accuracy. This concept is further demonstrated in Example 7.

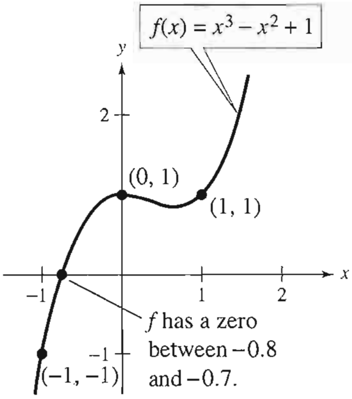


FIGURE 3.26

Example 7 ▶ Approximating a Zero of a Polynomial Function

Use the Intermediate Value Theorem to approximate the real zero of

$$f(x) = x^3 - x^2 + 1.$$

Solution

Begin by computing a few function values, as follows.

x	$f(x)$
-2	-11
-1	-1
0	1
1	1

Because $f(-1)$ is negative and $f(0)$ is positive, you can apply the Intermediate Value Theorem to conclude that the function has a zero between -1 and 0. To pinpoint this zero more closely, divide the interval $[-1, 0]$ into tenths and evaluate the function at each point. When you do this, you will find that

$$f(-0.8) = -0.152 \qquad \text{and} \qquad f(-0.7) = 0.167.$$

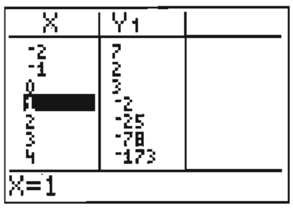
So, f must have a zero between -0.8 and -0.7, as shown in Figure 3.26. For a more accurate approximation, compute function values between $f(-0.8)$ and $f(-0.7)$ and apply the Intermediate Value Theorem again. By continuing this process, you can approximate this zero to any desired accuracy.

Technology

You can use the *table* feature of a graphing utility to approximate the zeros of a polynomial function. For instance, for the function

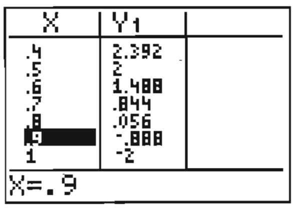
$$f(x) = -2x^3 - 3x^2 + 3$$

create a table that shows the function values for $-20 \le x \le 20$, as shown in the table above. Scroll through the table looking for consecutive function values that differ in sign. From the table above, you can see that $f(0)$ and $f(1)$ differ in sign. So, you can conclude from the Intermediate Value Theorem that the function has a zero between 0 and 1. You can adjust your table to show function values for $0 \le x \le 1$ using increments of 0.1, as shown below. By scrolling through the table on the right, you can see that $f(0.8)$ and $f(0.9)$ differ in sign. So, the function has a zero between 0.8 and 0.9. If you repeat this process several times, you should obtain $x \approx 0.806$ as the zero of the function. Use the *zero* or *root* feature of a graphing utility to confirm this result.



X	Y1
-2	7
-1	2
0	3
1	-2
2	-25
3	-78
4	-173

X=1



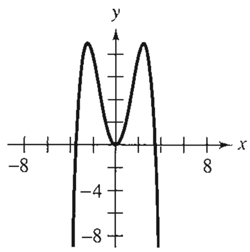
X	Y1
.4	2.392
.5	2
.6	1.488
.7	.844
.8	.056
.9	-.888
1	-2

X=.9

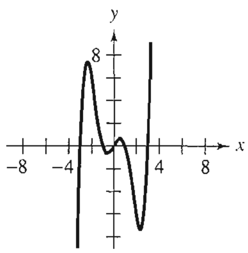
3.2 Exercises

In Exercises 1–8, match the polynomial function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

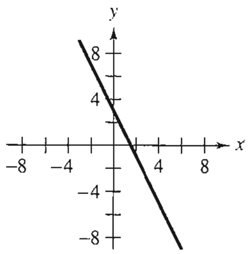
(a)



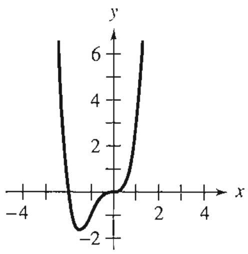
(b)



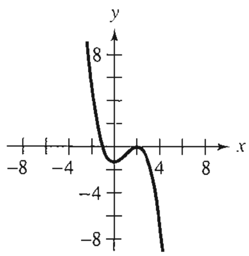
(c)



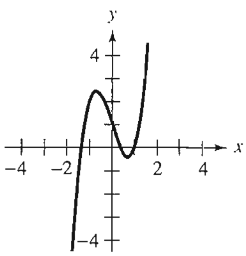
(d)



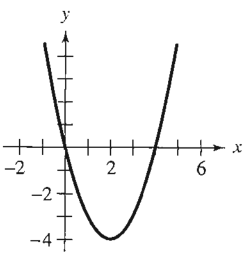
(e)



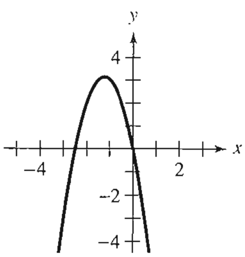
(f)



(g)



(h)



1. $f(x) = -2x + 3$
2. $f(x) = x^2 - 4x$
3. $f(x) = -2x^2 - 5x$
4. $f(x) = 2x^3 - 3x + 1$
5. $f(x) = -\frac{1}{4}x^4 + 3x^2$
6. $f(x) = -\frac{1}{3}x^3 + x^2 - \frac{4}{3}$
7. $f(x) = x^4 + 2x^3$
8. $f(x) = \frac{1}{5}x^5 - 2x^3 + \frac{9}{5}x$

In Exercises 9–12, sketch the graph of $y = x^n$ and each transformation.

9. $y = x^3$

(a) $f(x) = (x - 2)^3$ (b) $f(x) = x^3 - 2$

(c) $f(x) = -\frac{1}{2}x^3$ (d) $f(x) = (x - 2)^3 - 2$

10. $y = x^5$

(a) $f(x) = (x + 1)^5$ (b) $f(x) = x^5 + 1$

(c) $f(x) = 1 - \frac{1}{2}x^5$ (d) $f(x) = -\frac{1}{2}(x + 1)^5$

11. $y = x^4$

(a) $f(x) = (x + 3)^4$ (b) $f(x) = x^4 - 3$

(c) $f(x) = 4 - x^4$ (d) $f(x) = \frac{1}{2}(x - 1)^4$

(e) $f(x) = (2x)^4 + 1$ (f) $f(x) = \left(\frac{1}{2}x\right)^4 - 2$

12. $y = x^6$

(a) $f(x) = -\frac{1}{8}x^6$ (b) $f(x) = (x + 2)^6 - 4$

(c) $f(x) = x^6 - 4$ (d) $f(x) = -\frac{1}{4}x^6 + 1$

(e) $f(x) = \left(\frac{1}{4}x\right)^6 - 2$ (f) $f(x) = (2x)^6 - 1$

In Exercises 13–22, determine the right-hand and left-hand behavior of the graph of the polynomial function.

13. $f(x) = \frac{1}{3}x^3 + 5x$

14. $f(x) = 2x^2 - 3x + 1$

15. $g(x) = 5 - \frac{7}{2}x - 3x^2$

16. $h(x) = 1 - x^6$

17. $f(x) = -2.1x^5 + 4x^3 - 2$

18. $f(x) = 2x^5 - 5x + 7.5$

19. $f(x) = 6 - 2x + 4x^2 - 5x^3$

20. $f(x) = \dfrac{3x^4 - 2x + 5}{4}$

21. $h(t) = -\frac{2}{3}(t^2 - 5t + 3)$

22. $f(s) = -\frac{7}{8}(s^3 + 5s^2 - 7s + 1)$

Graphical Analysis In Exercises 23–26, use a graphing utility to graph the functions f and g in the same viewing window. Zoom out sufficiently far to show that the right-hand and left-hand behaviors of f and g appear identical.

23. $f(x) = 3x^3 - 9x + 1,\quad g(x) = 3x^3$

24. $f(x) = -\frac{1}{3}(x^3 - 3x + 2),\quad g(x) = -\frac{1}{3}x^3$

25. $f(x) = -(x^4 - 4x^3 + 16x),\quad g(x) = -x^4$

26. $f(x) = 3x^4 - 6x^2,\quad g(x) = 3x^4$

In Exercises 27–42, find all the real zeros of the polynomial function. Determine the multiplicity of each zero.

27. $f(x) = x^2 - 25$

28. $f(x) = 49 - x^2$

29. $h(t) = t^2 - 6t + 9$

30. $f(x) = x^2 + 10x + 25$

31. $f(x) = \frac{1}{3}x^2 + \frac{1}{3}x - \frac{2}{3}$

32. $f(x) = \frac{1}{2}x^2 + \frac{5}{2}x - \frac{3}{2}$

33. $f(x) = 3x^3 - 12x^2 + 3x$

34. $g(x) = 5x(x^2 - 2x - 1)$

35. $f(t) = t^3 - 4t^2 + 4t$

36. $f(x) = x^4 - x^3 - 20x^2$

37. $g(t) = t^5 - 6t^3 + 9t$

38. $f(x) = x^5 + x^3 - 6x$

39. $f(x) = 5x^4 + 15x^2 + 10$

40. $f(x) = 2x^4 - 2x^2 - 40$

41. $g(x) = x^3 + 3x^2 - 4x - 12$

42. $f(x) = x^3 - 4x^2 - 25x + 100$

Graphical Analysis In Exercises 43–46, use a graphing utility to graph the function. Use the graph to approximate any x-intercepts of the graph. Set $y = 0$ and solve the resulting equation. Compare the result with any x-intercepts of the graph.

43. $y = 4x^3 - 20x^2 + 25x$

44. $y = 4x^3 + 4x^2 - 7x + 2$

45. $y = x^5 - 5x^3 + 4x$

46. $y = \frac{1}{4}x^3(x^2 - 9)$

In Exercises 47–56, find a polynomial function that has the given zeros. (There are many correct answers.)

47. $0, 10$

48. $0, -3$

49. $2, -6$

50. $-4, 5$

51. $0, -2, -3$

52. $0, 2, 5$

53. $4, -3, 3, 0$

54. $-2, -1, 0, 1, 2$

55. $1 + \sqrt{3}, 1 - \sqrt{3}$

56. $2, 4 + \sqrt{5}, 4 - \sqrt{5}$

In Exercises 57–66, find a polynomial of degree n that has the given zero(s). (There are many correct answers.)

57. Zero: $x = -2$ Degree: $n = 2$

58. Zeros: $x = -8, -4$ Degree: $n = 2$

59. Zeros: $x = -3, 0, 1$ Degree: $n = 3$

60. Zeros: $x = -2, 4, 7$ Degree: $n = 3$

61. Zeros: $x = 0, \sqrt{3}, -\sqrt{3}$ Degree: $n = 3$

62. Zero: $x = 9$ Degree: $n = 3$

63. Zeros: $x = -5, 1, 2$ Degree: $n = 4$

64. Zeros: $x = -4, -1, 3, 6$ Degree: $n = 4$

65. Zeros: $x = 0, -4$ Degree: $n = 5$

66. Zeros: $x = -3, 1, 5, 6$ Degree: $n = 5$

In Exercises 67–80, sketch the graph of the function by (a) applying the Leading Coefficient Test, (b) finding the zeros of the polynomial, (c) plotting sufficient solution points, and (d) drawing a continuous curve through the points.

67. $f(x) = x^3 - 9x$

68. $g(x) = x^4 - 4x^2$

69. $f(t) = \frac{1}{4}(t^2 - 2t + 15)$

70. $g(x) = -x^2 + 10x - 16$

71. $f(x) = x^3 - 3x^2$

72. $f(x) = 1 - x^3$

73. $f(x) = 3x^3 - 15x^2 + 18x$

74. $f(x) = -4x^3 + 4x^2 + 15x$

75. $f(x) = -5x^2 - x^3$

76. $f(x) = -48x^2 + 3x^4$

77. $f(x) = x^2(x - 4)$

78. $h(x) = \frac{1}{3}x^3(x - 4)^2$

79. $g(t) = -\frac{1}{4}(t - 2)^2(t + 2)^2$

80. $g(x) = \frac{1}{10}(x + 1)^2(x - 3)^3$

In Exercises 81–84, use a graphing utility to graph the function. Use the *zero* or *root* feature to approximate zeros of the function. Determine the multiplicity of each zero.

81. $f(x) = x^3 - 4x$

82. $f(x) = \frac{1}{4}x^4 - 2x^2$

83. $g(x) = \frac{1}{5}(x + 1)^2(x - 3)(2x - 9)$

84. $h(x) = \frac{1}{5}(x + 2)^2(3x - 5)^2$

In Exercises 85–88, use the Intermediate Value Theorem and the *table* feature of a graphing utility to find intervals one unit in length in which the polynomial function is guaranteed to have a zero. Adjust the table to approximate the zeros of the function. Use the *zero* or *root* feature of a graphing utility to verify your results.

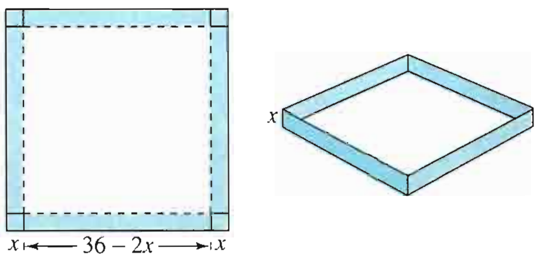
85. $f(x) = x^3 - 3x^2 + 3$

86. $f(x) = 0.11x^3 - 2.07x^2 + 9.81x - 6.88$

87. $g(x) = 3x^4 + 4x^3 - 3$

88. $h(x) = x^4 - 10x^2 + 3$

89. ***Numerical and Graphical Analysis*** An open box is to be made from a square piece of material, 36 inches on a side, by cutting equal squares with sides of length x from the corners and turning up the sides (see figure).



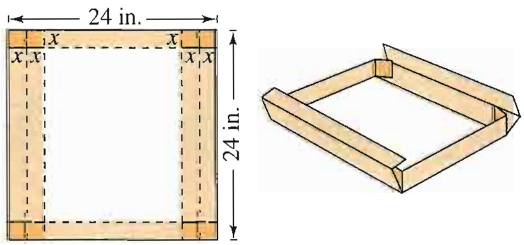
(a) Verify that the volume of the box is given by the function $V(x) = x(36 - 2x)^2$.

(b) Determine the domain of the function.

(c) Use a graphing utility to create a table that shows the box height x and the corresponding volumes V. Use the table to estimate the dimensions that will produce a maximum volume.

(d) Use a graphing utility to graph V and use the graph to estimate the value of x for which $V(x)$ is maximum. Compare your result with that of part (c).

90. ***Maximum Volume*** An open box with locking tabs is to be made from a square piece of material 24 inches on a side. This is to be done by cutting equal squares from the corners and folding along the dashed lines shown in the figure.



(a) Verify that the volume of the box is given by the function

$$V(x) = 8x(6 - x)(12 - x).$$

(b) Determine the domain of the function V.

(c) Sketch a graph of the function and estimate the value of x for which $V(x)$ is maximum.

▶ Model It

91. ***Tree Growth*** The growth of a red oak tree is approximated by the function

$$G = -0.003t^3 + 0.137t^2 + 0.458t - 0.839$$

where G is the height of the tree (in feet) and t $(2 \le t \le 34)$ is its age (in years).

(a) Use a graphing utility to graph the function. (*Hint:* Use a viewing window in which $-10 \le x \le 45$ and $-5 \le y \le 60$.)

(b) Estimate the age of the tree when it is growing most rapidly. This point is called the *point of diminishing returns* because the increase in size will be less with each additional year.

(c) Using calculus, the point of diminishing returns can also be found by finding the vertex of the parabola given by

$$y = -0.009t^2 + 0.274t + 0.458.$$

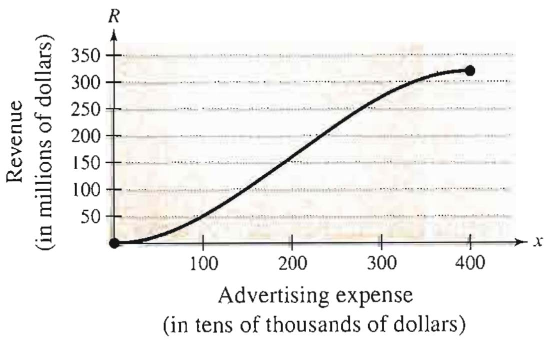
Find the vertex of this parabola.

(d) Compare your results from parts (b) and (c).

92. ***Revenue*** The total revenue R (in millions of dollars) for a company is related to its advertising expense by the function

$$R = \frac{1}{100{,}000}(-x^3 + 600x^2), \qquad 0 \le x \le 400$$

where x is the amount spent on advertising (in tens of thousands of dollars). Use the graph of this function, shown in the figure, to estimate the point on the graph at which the function is increasing most rapidly. This point is called the *point of diminishing returns* because any expense above this amount will yield less return per dollar invested in advertising.



Synthesis

True or False? **In Exercises 93–95, determine whether the statement is true or false. Justify your answer.**

93. A fifth-degree polynomial can have five turning points in its graph.

94. It is possible for a sixth-degree polynomial to have only one solution.

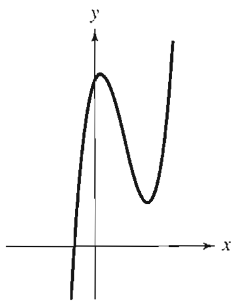
95. The graph of the function

$$f(x) = 2 + x - x^2 + x^3 - x^4 + x^5 + x^6 - x^7$$

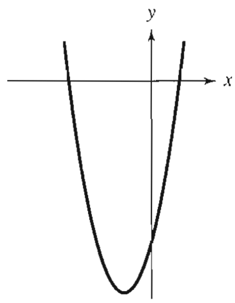
rises to the left and falls to the right.

96. ***Graphical Analysis*** Describe a polynomial function that could represent the graph. (Indicate the degree of the function and the sign of its leading coefficient.)

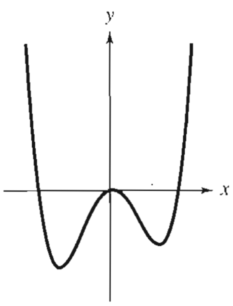
(a)



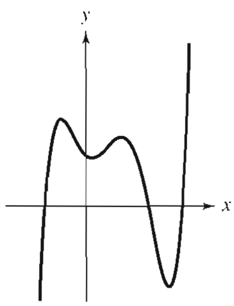
(b)



(c)



(d)



97. ***Graphical Reasoning*** Sketch a graph of the function $f(x) = x^4$. Explain how the graph of g differs (if it does) from the graph of f. Determine whether g is odd, even, or neither.

(a) $g(x) = f(x) + 2$ (b) $g(x) = f(x + 2)$

(c) $g(x) = f(-x)$ (d) $g(x) = -f(x)$

(e) $g(x) = f\left(\frac{1}{2}x\right)$ (f) $g(x) = \frac{1}{2}f(x)$

(g) $g(x) = f\left(x^{3/4}\right)$ (h) $g(x) = (f \circ f)(x)$

98. ***Exploration*** Explore the transformations of the form $g(x) = a(x - h)^5 + k$.

(a) Use a graphing utility to graph the functions

$$y_1 = -\frac{1}{3}(x - 2)^5 + 1$$

and

$$y_2 = \frac{3}{5}(x + 2)^5 - 3.$$

Determine whether the graphs are increasing or decreasing. Explain.

(b) Will the graph of g always be increasing or decreasing? If so, is this behavior determined by a, h, or k? Explain.

(c) Use a graphing utility to graph the function

$$H(x) = x^5 - 3x^3 + 2x + 1.$$

Use the graph and the result of part (b) to determine whether H can be written in the form $H(x) = a(x - h)^5 + k$. Explain.

Review

In Exercises 99–102, solve the equation by factoring.

99. $2x^2 - x - 28 = 0$ **100.** $3x^2 - 22x - 16 = 0$

101. $12x^2 + 11x - 5 = 0$ **102.** $x^2 + 24x + 144 = 0$

In Exercises 103–106, solve the equation by completing the square.

103. $x^2 - 2x - 21 = 0$ **104.** $x^2 - 8x + 2 = 0$

105. $2x^2 + 5x - 20 = 0$ **106.** $3x^2 + 4x - 9 = 0$

In Exercises 107–110, factor the expression completely.

107. $5x^2 + 7x - 24$ **108.** $6x^3 - 61x^2 + 10x$

109. $4x^4 - 7x^3 - 15x^2$ **110.** $y^3 + 216$

In Exercises 111–116, describe the transformation from a common function that occurs in the function. Then sketch its graph.

111. $f(x) = (x + 4)^2$ **112.** $f(x) = 3 - x^2$

113. $f(x) = \sqrt{x + 1} - 5$ **114.** $f(x) = 7 - \sqrt{x - 6}$

115. $f(x) = 2[\![x]\!] + 9$ **116.** $f(x) = 10 - \frac{1}{3}[\![x + 3]\!]$

3.3 Polynomial and Synthetic Division

▶ What you should learn

- How to use long division to divide polynomials by other polynomials
- How to use synthetic division to divide polynomials by binomials of the form $(x - k)$
- How to use the Remainder Theorem and the Factor Theorem

▶ Why you should learn it

Synthetic division can help you evaluate polynomial functions. For instance, in Exercise 75 on page 291, you will use synthetic division to determine the number of U.S. military personnel in 2005.

Kevin Fleming/Corbis

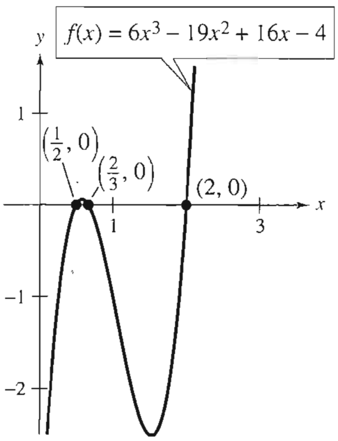


FIGURE 3.27

Long Division of Polynomials

In this section you will study two procedures for *dividing* polynomials. These procedures are especially valuable in factoring and finding the zeros of polynomial functions. To begin, suppose you are given the graph of

$$f(x) = 6x^3 - 19x^2 + 16x - 4.$$

Notice that a zero of f occurs at $x = 2$, as shown in Figure 3.27. Because $x = 2$ is a zero of f, you know that $(x - 2)$ is a factor of $f(x)$. This means that there exists a second-degree polynomial $q(x)$ such that

$$f(x) = (x - 2) \cdot q(x).$$

To find $q(x)$, you can use **long division,** as illustrated in Example 1.

Example 1 ▶ Long Division of Polynomials

Divide $6x^3 - 19x^2 + 16x - 4$ by $x - 2$, and use the result to factor the polynomial completely.

Solution

Think $\frac{6x^3}{x} = 6x^2$.

Think $\frac{-7x^2}{x} = -7x$.

Think $\frac{2x}{x} = 2$.

$$\begin{array}{rll}
 & 6x^2 - 7x + 2 & \\
x - 2 \,\big)\! & \overline{6x^3 - 19x^2 + 16x - 4} & \\
 & \underline{6x^3 - 12x^2} & \text{Multiply: } 6x^2(x - 2). \\
 & -7x^2 + 16x & \text{Subtract.} \\
 & \underline{-7x^2 + 14x} & \text{Multiply: } -7x(x - 2). \\
 & 2x - 4 & \text{Subtract.} \\
 & \underline{2x - 4} & \text{Multiply: } 2(x - 2). \\
 & 0 & \text{Subtract.}
\end{array}$$

From this division, you can conclude that

$$6x^3 - 19x^2 + 16x - 4 = (x - 2)(6x^2 - 7x + 2)$$

and by factoring the quadratic $6x^2 - 7x + 2$, you have

$$6x^3 - 19x^2 + 16x - 4 = (x - 2)(2x - 1)(3x - 2).$$

Note that this factorization agrees with the graph shown in Figure 3.27 in that the three x-intercepts occur at $x = 2$, $x = \frac{1}{2}$, and $x = \frac{2}{3}$.

In Example 1, $x - 2$ is a factor of the polynomial $6x^3 - 19x^2 + 16x - 4$, and the long division process produces a remainder of zero. Often, long division will produce a nonzero remainder. For instance, if you divide $x^2 + 3x + 5$ by $x + 1$, you obtain the following.

$$
\begin{array}{rrl}
 & x + 2 & \longleftarrow \text{Quotient} \\
\text{Divisor} \longrightarrow x + 1 & \overline{)\,x^2 + 3x + 5} & \longleftarrow \text{Dividend} \\
 & \underline{x^2 + x} & \\
 & 2x + 5 & \\
 & \underline{2x + 2} & \\
 & 3 & \longleftarrow \text{Remainder}
\end{array}
$$

In fractional form, you can write this result as follows.

$$\underbrace{\frac{\overbrace{x^2 + 3x + 5}^{\text{Dividend}}}{x + 1}}_{\text{Divisor}} = \overbrace{x + 2}^{\text{Quotient}} + \underbrace{\frac{\overset{\text{Remainder}}{3}}{x + 1}}_{\text{Divisor}}$$

This implies that

$$x^2 + 3x + 5 = (x + 1)(x + 2) + 3 \qquad \text{Multiply each side by } (x + 1).$$

which illustrates the following theorem, called the **Division Algorithm.**

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$\underset{\text{Dividend}}{f(x)} = \underset{\text{Divisor}}{d(x)}\underset{\text{Quotient}}{q(x)} + \underset{\text{Remainder}}{r(x)}$$

where $r(x) = 0$ *or* the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, $d(x)$ *divides evenly* into $f(x)$.

The Division Algorithm can also be written as

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

In the Division Algorithm, the rational expression $f(x)/d(x)$ is **improper** because the degree of $f(x)$ is greater than or equal to the degree of $d(x)$. On the other hand, the rational expression $r(x)/d(x)$ is **proper** because the degree of $r(x)$ is less than the degree of $d(x)$.

Before you apply the Division Algorithm, follow these steps.

1. Write the dividend and divisor in descending powers of the variable.
2. Insert placeholders with zero coefficients for missing powers of the variable.

Example 2 ▶ Long Division of Polynomials

Divide $x^3 - 1$ by $x - 1$.

Solution

Because there is no x^2-term or x-term in the dividend, you need to line up the subtraction by using zero coefficients (or leaving spaces) for the missing terms.

$$\begin{array}{r@{}l}
 & \quad x^2 + x + 1 \\
x - 1 & \overline{)\, x^3 + 0x^2 + 0x - 1} \\
 & \underline{x^3 - x^2} \\
 & \qquad x^2 + 0x \\
 & \qquad \underline{x^2 - x} \\
 & \qquad\qquad x - 1 \\
 & \qquad\qquad \underline{x - 1} \\
 & \qquad\qquad\qquad 0
\end{array}$$

So, $x - 1$ divides evenly into $x^3 - 1$, and you can write

$$\frac{x^3 - 1}{x - 1} = x^2 + x + 1, \quad x \neq 1.$$

You can check the result of Example 2 by multiplying.

$$(x - 1)(x^2 + x + 1) = x^3 + x^2 + x - x^2 - x - 1 = x^3 - 1$$

Example 3 ▶ Long Division of Polynomials

Divide $2x^4 + 4x^3 - 5x^2 + 3x - 2$ by $x^2 + 2x - 3$.

Solution

$$\begin{array}{r@{}l}
 & \qquad\quad 2x^2 \qquad\quad + 1 \\
x^2 + 2x - 3 & \overline{)\, 2x^4 + 4x^3 - 5x^2 + 3x - 2} \\
 & \underline{2x^4 + 4x^3 - 6x^2} \\
 & \qquad\qquad\quad x^2 + 3x - 2 \\
 & \qquad\qquad\quad \underline{x^2 + 2x - 3} \\
 & \qquad\qquad\qquad\qquad x + 1
\end{array}$$

Note that the first subtraction eliminated two terms from the dividend. When this happens, the quotient skips a term. You can write the result as

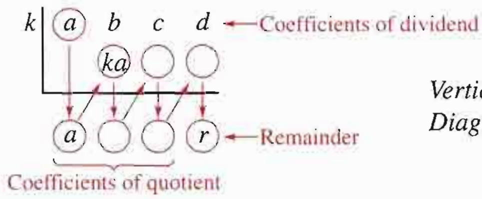
$$\frac{2x^4 + 4x^3 - 5x^2 + 3x - 2}{x^2 + 2x - 3} = 2x^2 + 1 + \frac{x + 1}{x^2 + 2x - 3}.$$

Synthetic Division

There is a nice shortcut for long division of polynomials when dividing by divisors of the form $x - k$. This shortcut is called **synthetic division.** The pattern for synthetic division of a cubic polynomial is summarized as follows. (The pattern for higher-degree polynomials is similar.)

Synthetic Division (for a Cubic Polynomial)

To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use the following pattern.



Vertical pattern: Add terms.
Diagonal pattern: Multiply by k.

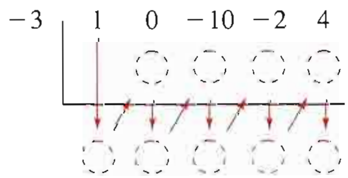
Synthetic division works only for divisors of the form $x - k$. [Remember that $x + k = x - (-k)$.] You cannot use synthetic division to divide a polynomial by a quadratic such as $x^2 - 3$.

Example 4 ▶ Using Synthetic Division

Use synthetic division to divide $x^4 - 10x^2 - 2x + 4$ by $x + 3$.

Solution

You should set up the array as follows. Note that a zero is included for the missing x^3-term in the dividend.



Then, use the synthetic division pattern by adding terms in columns and multiplying the results by -3.

Divisor: $x + 3$ Dividend: $x^4 - 10x^2 - 2x + 4$

$$
\begin{array}{r|rrrrr}
-3 & 1 & 0 & -10 & -2 & 4 \\
 & & -3 & 9 & 3 & -3 \\
\hline
 & 1 & -3 & -1 & 1 & 1
\end{array}
$$

Remainder: 1

Quotient: $x^3 - 3x^2 - x + 1$

So, you have

$$\frac{x^4 - 10x^2 - 2x + 4}{x + 3} = x^3 - 3x^2 - x + 1 + \frac{1}{x + 3}.$$

The Remainder and Factor Theorems

The remainder obtained in the synthetic division process has an important interpretation, as described in the **Remainder Theorem.**

> **The Remainder Theorem**
>
> If a polynomial $f(x)$ is divided by $x - k$, the remainder is
>
> $$r = f(k).$$

For a proof of the Remainder Theorem, see Proofs in Mathematics on page 326.

The Remainder Theorem tells you that synthetic division can be used to evaluate a polynomial function. That is, to evaluate a polynomial function $f(x)$ when $x = k$, divide $f(x)$ by $x - k$. The remainder will be $f(k)$, as illustrated in Example 5.

Example 5 ▶ Using the Remainder Theorem

Use the Remainder Theorem to evaluate the following function at $x = -2$.

$$f(x) = 3x^3 + 8x^2 + 5x - 7$$

Solution

Using synthetic division, you obtain the following.

$$\begin{array}{r|rrrr} -2 & 3 & 8 & 5 & -7 \\ & & -6 & -4 & -2 \\ \hline & 3 & 2 & 1 & -9 \end{array}$$

Because the remainder is $r = -9$, you can conclude that

$$f(-2) = -9. \qquad r = f(k)$$

This means that $(-2, -9)$ is a point on the graph of f. You can check this by substituting $x = -2$ in the original function.

Check

$$f(-2) = 3(-2)^3 + 8(-2)^2 + 5(-2) - 7$$
$$= 3(-8) + 8(4) - 10 - 7 = -9$$

Another important theorem is the **Factor Theorem,** stated below. This theorem states that you can test to see whether a polynomial has $(x - k)$ as a factor by evaluating the polynomial at $x = k$. If the result is 0, $(x - k)$ is a factor.

> **The Factor Theorem**
>
> A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

For a proof of the Factor Theorem, see Proofs in Mathematics on page 326.

Example 6 ▶ Factoring a Polynomial: Repeated Division

Show that $(x - 2)$ and $(x + 3)$ are factors of

$$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18.$$

Then find the remaining factors of $f(x)$.

Solution

Using synthetic division with the factor $(x - 2)$, you obtain the following.

$$\begin{array}{r|rrrrr} 2 & 2 & 7 & -4 & -27 & -18 \\ & & 4 & 22 & 36 & 18 \\ \hline & 2 & 11 & 18 & 9 & 0 \end{array}$$

0 remainder, so $f(2) = 0$ and $(x - 2)$ is a factor.

Take the result of this division and perform synthetic division again using the factor $(x + 3)$.

$$\begin{array}{r|rrrr} -3 & 2 & 11 & 18 & 9 \\ & & -6 & -15 & -9 \\ \hline & 2 & 5 & 3 & 0 \end{array}$$

0 remainder, so $f(-3) = 0$ and $(x + 3)$ is a factor.

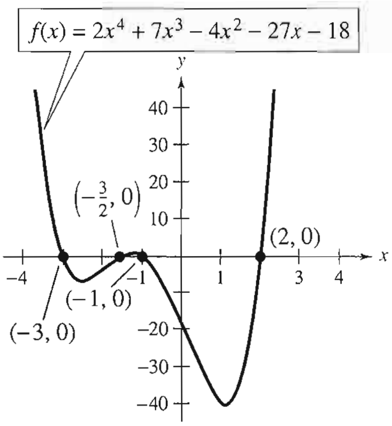


FIGURE 3.28

Because the resulting quadratic expression factors as

$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

the complete factorization of $f(x)$ is

$$f(x) = (x - 2)(x + 3)(2x + 3)(x + 1).$$

Note that this factorization implies that f has four real zeros:

$$x = 2,\ x = -3,\ x = -\tfrac{3}{2}, \text{ and } x = -1.$$

This is confirmed by the graph of f, which is shown in Figure 3.28.

Uses of the Remainder in Synthetic Division

The remainder r, obtained in the synthetic division of $f(x)$ by $x - k$, provides the following information.

1. The remainder r gives the value of f at $x = k$. That is, $r = f(k)$.
2. If $r = 0$, $(x - k)$ is a factor of $f(x)$.
3. If $r = 0$, $(k, 0)$ is an x-intercept of the graph of f.

Throughout this text, the importance of developing several problem-solving strategies is emphasized. In the exercises for this section, try using more than one strategy to solve several of the exercises. For instance, if you find that $x - k$ divides evenly into $f(x)$ (with no remainder), try sketching the graph of f. You should find that $(k, 0)$ is an x-intercept of the graph.

3.3 Exercises

Analytical Analysis In Exercises 1 and 2, use long division to verify that $y_1 = y_2$.

1. $y_1 = \dfrac{x^2}{x+2}$, $y_2 = x - 2 + \dfrac{4}{x+2}$

2. $y_1 = \dfrac{x^4 - 3x^2 - 1}{x^2 + 5}$, $y_2 = x^2 - 8 + \dfrac{39}{x^2+5}$

Graphical Analysis In Exercises 3 and 4, use a graphing utility to graph the two equations in the same viewing window. Use the graphs to verify that the expressions are equivalent. Use long division to verify the results algebraically.

3. $y_1 = \dfrac{x^5 - 3x^3}{x^2 + 1}$, $y_2 = x^3 - 4x + \dfrac{4x}{x^2+1}$

4. $y_1 = \dfrac{x^3 - 2x^2 + 5}{x^2 + x + 1}$, $y_2 = x - 3 + \dfrac{2(x+4)}{x^2+x+1}$

In Exercises 5–18, use long division to divide.

5. $(2x^2 + 10x + 12) \div (x + 3)$
6. $(5x^2 - 17x - 12) \div (x - 4)$
7. $(4x^3 - 7x^2 - 11x + 5) \div (4x + 5)$
8. $(6x^3 - 16x^2 + 17x - 6) \div (3x - 2)$
9. $(x^4 + 5x^3 + 6x^2 - x - 2) \div (x + 2)$
10. $(x^3 + 4x^2 - 3x - 12) \div (x - 3)$
11. $(7x + 3) \div (x + 2)$
12. $(8x - 5) \div (2x + 1)$
13. $(6x^3 + 10x^2 + x + 8) \div (2x^2 + 1)$
14. $(x^3 - 9) \div (x^2 + 1)$
15. $\dfrac{x^4 + 3x^2 + 1}{x^2 - 2x + 3}$
16. $\dfrac{x^5 + 7}{x^3 - 1}$
17. $\dfrac{x^4}{(x-1)^3}$
18. $\dfrac{2x^3 - 4x^2 - 15x + 5}{(x-1)^2}$

In Exercises 19–36, use synthetic division to divide.

19. $(3x^3 - 17x^2 + 15x - 25) \div (x - 5)$
20. $(5x^3 + 18x^2 + 7x - 6) \div (x + 3)$
21. $(4x^3 - 9x + 8x^2 - 18) \div (x + 2)$
22. $(9x^3 - 16x - 18x^2 + 32) \div (x - 2)$
23. $(-x^3 + 75x - 250) \div (x + 10)$
24. $(3x^3 - 16x^2 - 72) \div (x - 6)$
25. $(5x^3 - 6x^2 + 8) \div (x - 4)$
26. $(5x^3 + 6x + 8) \div (x + 2)$
27. $\dfrac{10x^4 - 50x^3 - 800}{x - 6}$
28. $\dfrac{x^5 - 13x^4 - 120x + 80}{x + 3}$
29. $\dfrac{x^3 + 512}{x + 8}$
30. $\dfrac{x^3 - 729}{x - 9}$
31. $\dfrac{-3x^4}{x - 2}$
32. $\dfrac{-3x^4}{x + 2}$
33. $\dfrac{180x - x^4}{x - 6}$
34. $\dfrac{5 - 3x + 2x^2 - x^3}{x + 1}$
35. $\dfrac{4x^3 + 16x^2 - 23x - 15}{x + \frac{1}{2}}$
36. $\dfrac{3x^3 - 4x^2 + 5}{x - \frac{3}{2}}$

In Exercises 37–44, express the function in the form $f(x) = (x - k)q(x) + r$ for the given value of k, and demonstrate that $f(k) = r$.

	Function	*Value of k*
37.	$f(x) = x^3 - x^2 - 14x + 11$	$k = 4$
38.	$f(x) = x^3 - 5x^2 - 11x + 8$	$k = -2$
39.	$f(x) = 15x^4 + 10x^3 - 6x^2 + 14$	$k = -\frac{2}{3}$
40.	$f(x) = 10x^3 - 22x^2 - 3x + 4$	$k = \frac{1}{5}$
41.	$f(x) = x^3 + 3x^2 - 2x - 14$	$k = \sqrt{2}$
42.	$f(x) = x^3 + 2x^2 - 5x - 4$	$k = -\sqrt{5}$
43.	$f(x) = -4x^3 + 6x^2 + 12x + 4$	$k = 1 - \sqrt{3}$
44.	$f(x) = -3x^3 + 8x^2 + 10x - 8$	$k = 2 + \sqrt{2}$

In Exercises 45–48, use synthetic division to find each function value. Verify your answers using another method.

45. $f(x) = 4x^3 - 13x + 10$
 (a) $f(1)$ (b) $f(-2)$ (c) $f\left(\frac{1}{2}\right)$ (d) $f(8)$
46. $g(x) = x^6 - 4x^4 + 3x^2 + 2$
 (a) $g(2)$ (b) $g(-4)$ (c) $g(3)$ (d) $g(-1)$
47. $h(x) = 3x^3 + 5x^2 - 10x + 1$
 (a) $h(3)$ (b) $h\left(\frac{1}{3}\right)$ (c) $h(-2)$ (d) $h(-5)$
48. $f(x) = 0.4x^4 - 1.6x^3 + 0.7x^2 - 2$
 (a) $f(1)$ (b) $f(-2)$ (c) $f(5)$ (d) $f(-10)$

In Exercises 49–56, use synthetic division to show that x is a zero of the third-degree polynomial equation, and use the result to factor the polynomial completely. List all real zeros of the function.

Polynomial Equation	*Value of x*
49. $x^3 - 7x + 6 = 0$	$x = 2$
50. $x^3 - 28x - 48 = 0$	$x = -4$
51. $2x^3 - 15x^2 + 27x - 10 = 0$	$x = \frac{1}{2}$
52. $48x^3 - 80x^2 + 41x - 6 = 0$	$x = \frac{2}{3}$
53. $x^3 + 2x^2 - 3x - 6 = 0$	$x = \sqrt{3}$
54. $x^3 + 2x^2 - 2x - 4 = 0$	$x = \sqrt{2}$
55. $x^3 - 3x^2 + 2 = 0$	$x = 1 + \sqrt{3}$
56. $x^3 - x^2 - 13x - 3 = 0$	$x = 2 - \sqrt{5}$

In Exercises 57–64, (a) verify the given factors of the function f, (b) find the remaining factors of f, (c) use your results to write the complete factorization of f, (d) list all real zeros of f, and (e) confirm your results by using a graphing utility to graph the function.

Function	*Factors*
57. $f(x) = 2x^3 + x^2 - 5x + 2$	$(x + 2), (x - 1)$
58. $f(x) = 3x^3 + 2x^2 - 19x + 6$	$(x + 3), (x - 2)$
59. $f(x) = x^4 - 4x^3 - 15x^2 + 58x - 40$	$(x - 5), (x + 4)$
60. $f(x) = 8x^4 - 14x^3 - 71x^2 - 10x + 24$	$(x + 2), (x - 4)$
61. $f(x) = 6x^3 + 41x^2 - 9x - 14$	$(2x + 1), (3x - 2)$
62. $f(x) = 10x^3 - 11x^2 - 72x + 45$	$(2x + 5), (5x - 3)$
63. $f(x) = 2x^3 - x^2 - 10x + 5$	$(2x - 1), (x + \sqrt{5})$
64. $f(x) = x^3 + 3x^2 - 48x - 144$	$(x + 4\sqrt{3}), (x + 3)$

Graphical Analysis **In Exercises 65–68, (a) use the *zero* or *root* feature of a graphing utility to approximate the zeros of the function accurate to three decimal places and (b) determine one of the exact zeros, use synthetic division to verify your result, and then factor the polynomial completely.**

65. $f(x) = x^3 - 2x^2 - 5x + 10$

66. $g(x) = x^3 - 4x^2 - 2x + 8$

67. $h(t) = t^3 - 2t^2 - 7t + 2$

68. $f(s) = s^3 - 12s^2 + 40s - 24$

In Exercises 69–74, simplify the rational expression.

69. $\dfrac{4x^3 - 8x^2 + x + 3}{2x - 3}$

70. $\dfrac{x^3 + x^2 - 64x - 64}{x + 8}$

71. $\dfrac{x^3 + 3x^2 - x - 3}{x + 1}$

72. $\dfrac{2x^3 + 3x^2 - 3x - 2}{x - 1}$

73. $\dfrac{x^4 + 6x^3 + 11x^2 + 6x}{x^2 + 3x + 2}$

74. $\dfrac{x^4 + 9x^3 - 5x^2 - 36x + 4}{x^2 - 4}$

▶ Model It

75. ***Data Analysis*** The numbers M (in thousands) of United States military personnel on active duty for the years 1990 through 2000 are shown in the table, where t represents the time (in years), with $t = 0$ corresponding to 1990. (Source: U.S. Department of Defense)

Year, t	Military personnel, M
0	2044
1	1986
2	1807
3	1705
4	1610
5	1518
6	1472
7	1439
8	1407
9	1386
10	1384

(a) Use a graphing utility to create a scatter plot of the data.

(b) Use the *regression* feature of the graphing utility to find a cubic model for the data. Then graph the model in the same viewing window as the scatter plot.

(c) Use the model to create a table of estimated values of M. Compare the model with the data.

(d) Use synthetic division to evaluate the model for the year 2005. Even though the model is relatively accurate for estimating the given data, would you use this model to predict the number of military personnel in the future? Explain.

76. ***Data Analysis*** The average monthly basic rates R (in dollars) for cable television in the United States for the years 1990 through 1999 are shown in the table, where t represents the time (in years), with $t = 0$ corresponding to 1990. (Source: Paul Kagan Associates, Inc.)

Year, t	Basic rate, R
0	16.78
1	18.10
2	19.08
3	19.39
4	21.62
5	23.07
6	24.41
7	26.48
8	27.81
9	28.92

(a) Use a graphing utility to create a scatter plot of the data.

(b) Use the *regression* feature of the graphing utility to find a cubic model for the data. Then graph the model in the same viewing window as the scatter plot. Compare the model with the data.

(c) Use synthetic division to evaluate the model for the year 2005.

Synthesis

True or False? **In Exercises 77–79, determine whether the statement is true or false. Justify your answer.**

77. If $(7x + 4)$ is a factor of some polynomial function f, then $\frac{4}{7}$ is a zero of f.

78. $(2x - 1)$ is a factor of the polynomial $6x^6 + x^5 - 92x^4 + 45x^3 + 184x^2 + 4x - 48$.

79. The rational expression

$$\frac{x^3 + 2x^2 - 13x + 10}{x^2 - 4x - 12}$$

is improper.

80. ***Exploration*** Use the form

$$f(x) = (x - k)q(x) + r$$

to create a cubic function that (a) passes through the point $(2, 5)$ and rises to the right, and (b) passes through the point $(-3, 1)$ and falls to the right. (There are many correct answers.)

Think About It **In Exercises 81 and 82, perform the division by assuming that n is a positive integer.**

81. $\dfrac{x^{3n} + 9x^{2n} + 27x^n + 27}{x^n + 3}$

82. $\dfrac{x^{3n} - 3x^{2n} + 5x^n - 6}{x^n - 2}$

83. ***Writing*** Briefly explain what it means for a divisor to divide evenly into a dividend.

84. ***Writing*** Briefly explain how to check polynomial division, and justify your reasoning. Give an example.

Exploration **In Exercises 85 and 86, find the constant c such that the denominator will divide evenly into the numerator.**

85. $\dfrac{x^3 + 4x^2 - 3x + c}{x - 5}$

86. $\dfrac{x^5 - 2x^2 + x + c}{x + 2}$

Think About It **In Exercises 87 and 88, answer the questions about the division**

$$\frac{f(x)}{x - k}$$

where $f(x) = (x + 3)^2(x - 3)(x + 1)^3$.

87. What is the remainder when $k = -3$? Explain.

88. If it is necessary to find $f(2)$, is it easier to evaluate the function directly or to use synthetic division? Explain.

Review

In Exercises 89–94, use any method to solve the quadratic equation.

89. $9x^2 - 25 = 0$

90. $16x^2 - 21 = 0$

91. $5x^2 - 3x - 14 = 0$

92. $8x^2 - 22x + 15 = 0$

93. $2x^2 + 6x + 3 = 0$

94. $x^2 + 3x - 3 = 0$

In Exercises 95–98, find a polynomial function that has the given zeros. There are many correct answers.

95. $0, 3, 4$

96. $-6, 1$

97. $-3, 1 + \sqrt{2}, 1 - \sqrt{2}$

98. $1, -2, 2 + \sqrt{3}, 2 - \sqrt{3}$

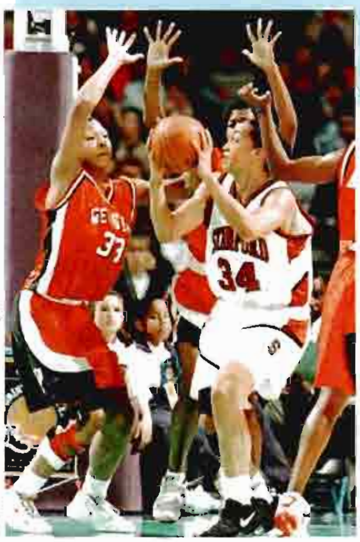
3.4 Zeros of Polynomial Functions

▶ **What you should learn**

- How to use the Fundamental Theorem of Algebra to determine the number of zeros of polynomial functions
- How to find rational zeros of polynomial functions
- How to find conjugate pairs of complex zeros
- How to find zeros of polynomials by factoring
- How to use Descartes's Rule of Signs and the Upper and Lower Bound Rules to find zeros of polynomials

▶ **Why you should learn it**

Finding zeros of polynomial functions is an important part of solving real-life problems. For instance, in Exercise 107 on page 306, the zeros of a polynomial function can help you analyze the attendance at women's college basketball games.



Jonathan Daniel/Getty Images

The Fundamental Theorem of Algebra

You know that an nth-degree polynomial can have at most n real zeros. In the complex number system, this statement can be improved. That is, in the complex number system, every nth-degree polynomial function has *precisely* n zeros. This important result is derived from the **Fundamental Theorem of Algebra,** first proved by the German mathematician Carl Friedrich Gauss (1777–1855).

The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.

Using the Fundamental Theorem of Algebra and the equivalence of zeros and factors, you obtain the **Linear Factorization Theorem.**

Linear Factorization Theorem

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers.

For a proof of the Linear Factorization Theorem, see Proofs in Mathematics on page 327.

Note that the Fundamental Theorem of Algebra and the Linear Factorization Theorem tell you only that the zeros or factors of a polynomial exist, not how to find them. Such theorems are called *existence theorems*.

Example 1 ▶ Zeros of Polynomial Functions

a. The first-degree polynomial $f(x) = x - 2$ has exactly *one* zero: $x = 2$.

b. Counting multiplicity, the second-degree polynomial function

$$f(x) = x^2 - 6x + 9 = (x - 3)(x - 3)$$

has exactly *two* zeros: $x = 3$ and $x = 3$. (This is called a *repeated zero*.)

c. The third-degree polynomial function

$$f(x) = x^3 + 4x = x(x^2 + 4) = x(x - 2i)(x + 2i)$$

has exactly *three* zeros: $x = 0$, $x = 2i$, and $x = -2i$.

d. The fourth-degree polynomial function

$$f(x) = x^4 - 1 = (x - 1)(x + 1)(x - i)(x + i)$$

has exactly *four* zeros: $x = 1$, $x = -1$, $x = i$, and $x = -i$.

The Rational Zero Test

The **Rational Zero Test** relates the possible rational zeros of a polynomial (having integer coefficients) to the leading coefficient and to the constant term of the polynomial.

Historical Note

Although they were not contemporaries, Jean Le Rond d'Alembert (1717–1783) worked independently of Carl Gauss in trying to prove the Fundamental Theorem of Algebra. His efforts were such that, in France, the Fundamental Theorem of Algebra is frequently known as the Theorem of d'Alembert.

The Rational Zero Test

If the polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$ has *integer* coefficients, every rational zero of f has the form

$$\text{Rational zero} = \frac{p}{q}$$

where p and q have no common factors other than 1, and

$p =$ a factor of the constant term a_0

$q =$ a factor of the leading coefficient a_n.

To use the Rational Zero Test, you should first list all rational numbers whose numerators are factors of the constant term and whose denominators are factors of the leading coefficient.

$$\text{Possible rational zeros} = \frac{\text{factors of constant term}}{\text{factors of leading coefficient}}$$

Having formed this list of *possible rational zeros*, use a trial-and-error method to determine which, if any, are actual zeros of the polynomial. Note that when the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term.

Example 2 ▶ Rational Zero Test with Leading Coefficient of 1

Find the rational zeros of

$$f(x) = x^3 + x + 1.$$

Solution

Because the leading coefficient is 1, the possible rational zeros are ± 1, the factors of the constant term. By testing these possible zeros, you can see that neither works.

$$f(1) = (1)^3 + 1 + 1$$
$$= 3$$
$$f(-1) = (-1)^3 + (-1) + 1$$
$$= -1$$

So, you can conclude that the given polynomial has *no* rational zeros. Note from the graph of f in Figure 3.29 that f does have one real zero between -1 and 0. However, by the Rational Zero Test, you know that this real zero is *not* a rational number.

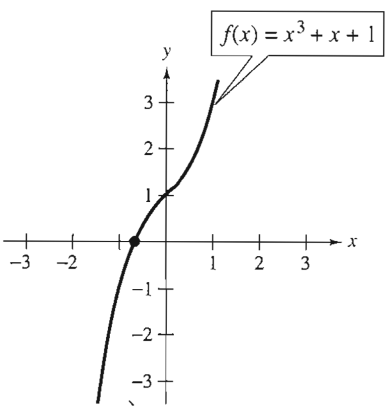


FIGURE 3.29

Example 3 ▶ Rational Zero Test with Leading Coefficient of 1

Find the rational zeros of $f(x) = x^4 - x^3 + x^2 - 3x - 6$.

Solution

Because the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

Possible rational zeros: $\pm 1, \pm 2, \pm 3, \pm 6$

A test of these possible zeros shows that $x = -1$ and $x = 2$ are the only two rational zeros. Check the others to be sure.

If the leading coefficient of a polynomial is not 1, the list of possible rational zeros can increase dramatically. In such cases, the search can be shortened in several ways: (1) a programmable calculator can be used to speed up the calculations; (2) a graph, drawn either by hand or with a graphing utility, can give a good estimate of the locations of the zeros; (3) the Intermediate Value Theorem along with a table generated by a graphing utility can give approximations of zeros; and (4) synthetic division can be used to test the possible rational zeros.

To see how to use synthetic division to test the possible rational zeros, take another look at the function $f(x) = x^4 - x^3 + x^2 - 3x - 6$ from Example 3. To test that $x = -1$ and $x = 2$ are zeros of f, you can apply synthetic division successively, as follows.

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & 1 & -3 & -6 \\ & & -1 & 2 & -3 & 6 \\ \hline & 1 & -2 & 3 & -6 & 0 \end{array}$$

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 3 & -6 \\ & & 2 & 0 & 6 \\ \hline & 1 & 0 & 3 & 0 \end{array}$$

So, you have

$$f(x) = (x + 1)(x - 2)(x^2 + 3).$$

Because the factor $(x^2 + 3)$ produces no real zeros, you can conclude that $x = -1$ and $x = 2$ are the only *real* zeros of f, which is verified in Figure 3.30.

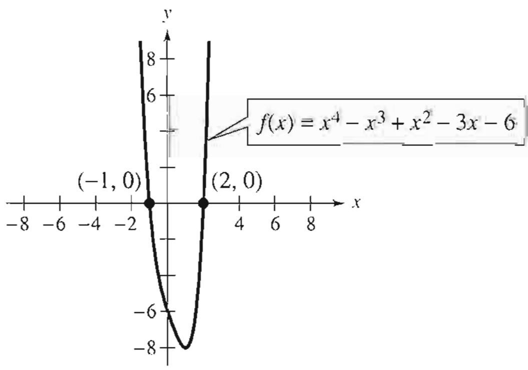


FIGURE 3.30

Finding the first zero is often the hardest part. After that, the search is simplified by using the lower-degree polynomial obtained in synthetic division.

Example 4 ▶ Using the Rational Zero Test

Find the rational zeros of $f(x) = 2x^3 + 3x^2 - 8x + 3$.

Solution

The leading coefficient is 2 and the constant term is 3.

Possible rational zeros: $\dfrac{\text{Factors of } 3}{\text{Factors of } 2} = \dfrac{\pm 1, \pm 3}{\pm 1, \pm 2} = \pm 1, \pm 3, \pm \dfrac{1}{2}, \pm \dfrac{3}{2}$

By synthetic division, you can determine that $x = 1$ is a rational zero.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

So, $f(x)$ factors as

$$\begin{aligned} f(x) &= (x - 1)(2x^2 + 5x - 3) \\ &= (x - 1)(2x - 1)(x + 3) \end{aligned}$$

and you can conclude that the rational zeros of f are $x = 1$, $x = \frac{1}{2}$, and $x = -3$.

Example 5 ▶ Using the Rational Zero Test

Find all the real zeros of $f(x) = -10x^3 + 15x^2 + 16x - 12$.

Solution

The leading coefficient is -10 and the constant term is -12.

Possible rational zeros: $\dfrac{\text{Factors of } -12}{\text{Factors of } -10} = \dfrac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1, \pm 2, \pm 5, \pm 10}$

With so many possibilities (32, in fact), it is worth your time to stop and sketch a graph. From Figure 3.31, it looks like three reasonable choices would be $x = -\frac{6}{5}$, $x = \frac{1}{2}$, and $x = 2$. Testing these by synthetic division shows that only $x = 2$ is a zero. So, you have

$$f(x) = (x - 2)(-10x^2 - 5x + 6).$$

Using the Quadratic Formula for the second factor, you find that the two additional zeros are irrational numbers.

$$x = \frac{-(-5) + \sqrt{265}}{-20} \approx -1.0639$$

and

$$x = \frac{-(-5) - \sqrt{265}}{-20} \approx 0.5639$$

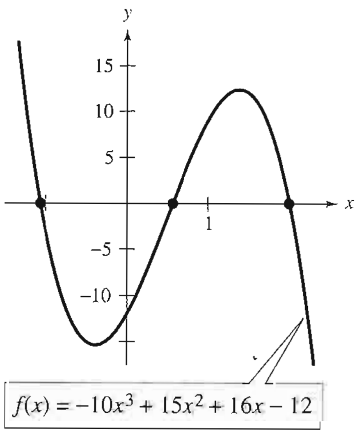


FIGURE 3.31

Conjugate Pairs

In Example 1(c) and (d), note that the pairs of complex zeros are **conjugates.** That is, they are of the form $a + bi$ and $a - bi$.

Complex Zeros Occur in Conjugate Pairs

Let $f(x)$ be a polynomial function that has *real coefficients*. If $a + bi$, where $b \neq 0$, is a zero of the function, the conjugate $a - bi$ is also a zero of the function.

Be sure you see that this result is true only if the polynomial function has *real coefficients*. For instance, the result applies to the function $f(x) = x^2 + 1$ but not to the function $g(x) = x - i$.

Example 6 ▶ Finding a Polynomial with Given Zeros

Find a fourth-degree polynomial function with real coefficients that has -1, -1, and $3i$ as zeros.

Solution

Because $3i$ is a zero *and* the polynomial is stated to have real coefficients, you know that the conjugate $-3i$ must also be a zero. So, from the Linear Factorization Theorem, $f(x)$ can be written as

$$f(x) = a(x + 1)(x + 1)(x - 3i)(x + 3i).$$

For simplicity, let $a = 1$ to obtain

$$\begin{aligned} f(x) &= (x^2 + 2x + 1)(x^2 + 9) \\ &= x^4 + 2x^3 + 10x^2 + 18x + 9. \end{aligned}$$

Factoring a Polynomial

The Linear Factorization Theorem shows that you can write any nth-degree polynomial as the product of n linear factors.

$$f(x) = a_n(x - c_1)(x - c_2)(x - c_3) \cdot \cdot \cdot (x - c_n)$$

However, this result includes the possibility that some of the values of c_i are complex. The following theorem says that even if you do not want to get involved with "complex factors," you can still write $f(x)$ as the product of linear and/or quadratic factors. For a proof of this theorem, see Proofs in Mathematics on page 327.

Factors of a Polynomial

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

A quadratic factor with no real zeros is said to be *prime* or **irreducible over the reals.** Be sure you see that this is not the same as being *irreducible over the rationals*. For example, the quadratic

$$x^2 + 1 = (x - i)(x + i)$$

is irreducible over the reals (and therefore over the rationals). On the other hand, the quadratic

$$x^2 - 2 = \left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)$$

is irreducible over the rationals but *reducible* over the reals.

Example 7 ▶ Finding the Zeros of a Polynomial Function

Find all the zeros of

$$f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$$

given that $1 + 3i$ is a zero of f.

Solution

Because complex zeros occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. This means that both

$$[x - (1 + 3i)] \quad \text{and} \quad [x - (1 - 3i)]$$

are factors of f. Multiplying these two factors produces

$$\begin{aligned}
[x - (1 + 3i)][x - (1 - 3i)] &= [(x - 1) - 3i][(x - 1) + 3i] \\
&= (x - 1)^2 - 9i^2 \\
&= x^2 - 2x + 1 - 9(-1) \\
&= x^2 - 2x + 10.
\end{aligned}$$

Using long division, you can divide $x^2 - 2x + 10$ into f to obtain the following.

$$\begin{array}{rrrrrr}
 & & x^2 & -\ x & -\ 6 & \\
x^2 - 2x + 10 \,\big) & x^4 & -\ 3x^3 & +\ 6x^2 & +\ 2x & -\ 60 \\
 & \underline{x^4} & \underline{-\ 2x^3} & \underline{+\ 10x^2} & & \\
 & & -x^3 & -\ 4x^2 & +\ 2x & \\
 & & \underline{-x^3} & \underline{+\ 2x^2} & \underline{-\ 10x} & \\
 & & & -6x^2 & +\ 12x & -\ 60 \\
 & & & \underline{-6x^2} & \underline{+\ 12x} & \underline{-\ 60} \\
 & & & & & 0
\end{array}$$

So, you have

$$\begin{aligned}
f(x) &= (x^2 - 2x + 10)(x^2 - x - 6) \\
&= (x^2 - 2x + 10)(x - 3)(x + 2)
\end{aligned}$$

and you can conclude that the zeros of f are $x = 1 + 3i$, $x = 1 - 3i$, $x = 3$, and $x = -2$.

STUDY TIP

In Example 7, if you were not told that $1 + 3i$ is a zero of f, you could still find all zeros of the function by using synthetic division to find the real zeros -2 and 3. Then you could factor the polynomial as $(x + 2)(x - 3)(x^2 - 2x + 10)$. Finally, by using the Quadratic Formula, you could determine that the zeros are $x = -2$, $x = 3$, $x = 1 + 3i$, and $x = 1 - 3i$.

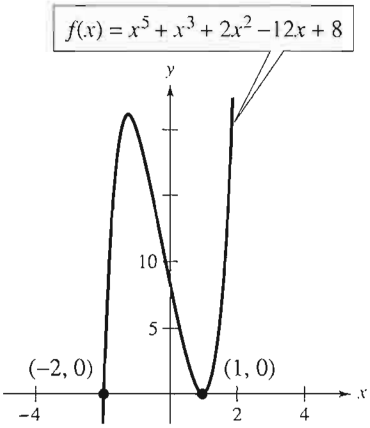


FIGURE 3.32

Example 8 shows how to find all the zeros of a polynomial function, including complex zeros.

Example 8 ▶ Finding the Zeros of a Polynomial Function

Write $f(x) = x^5 + x^3 + 2x^2 - 12x + 8$ as the product of linear factors, and list all of its zeros.

Solution

The possible rational zeros are ±1, ±2, ±4, and ±8. Synthetic division produces the following.

$$\begin{array}{r|rrrrrr} 1 & 1 & 0 & 1 & 2 & -12 & 8 \\ & & 1 & 1 & 2 & 4 & -8 \\ \hline & 1 & 1 & 2 & 4 & -8 & 0 \end{array} \longrightarrow \text{1 is a zero.}$$

$$\begin{array}{r|rrrrr} -2 & 1 & 1 & 2 & 4 & -8 \\ & & -2 & 2 & -8 & 8 \\ \hline & 1 & -1 & 4 & -4 & 0 \end{array} \longrightarrow \text{-2 is a zero.}$$

So, you have

$$\begin{aligned} f(x) &= x^5 + x^3 + 2x^2 - 12x + 8 \\ &= (x - 1)(x + 2)(x^3 - x^2 + 4x - 4). \end{aligned}$$

You can factor $x^3 - x^2 + 4x - 4$ as $(x - 1)(x^2 + 4)$, and by factoring $x^2 + 4$ as

$$\begin{aligned} x^2 - (-4) &= \left(x - \sqrt{-4}\right)\left(x + \sqrt{-4}\right) \\ &= (x - 2i)(x + 2i) \end{aligned}$$

you obtain

$$f(x) = (x - 1)(x - 1)(x + 2)(x - 2i)(x + 2i)$$

which gives the following five zeros of f.

$$x = 1,\ x = 1,\ x = -2,\ x = 2i, \quad \text{and} \quad x = -2i$$

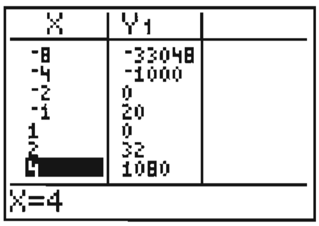
From the graph of f shown in Figure 3.32, you can see that the *real* zeros are the only ones that appear as x-intercepts. Note that $x = 1$ is a repeated zero.

STUDY TIP

In Example 8, the fifth-degree polynomial function has three real zeros. In such cases, you can use the *zoom* and *trace* features or the *zero* or *root* feature of a graphing utility to approximate the real zeros. You can then use these real zeros to determine the complex zeros algebraically.

Technology

You can use the *table* feature of a graphing utility to help you determine which of the possible rational zeros are zeros of the polynomial in Example 8. The table should be set to *ask* mode. Then enter each of the possible rational zeros in the table. When you do this, you will see that there are two rational zeros, −2 and 1, as shown at the right.



Other Tests for Zeros of Polynomials

You know that an nth-degree polynomial function can have *at most n* real zeros. Of course, many nth-degree polynomials do not have that many real zeros. For instance, $f(x) = x^2 + 1$ has no real zeros, and $f(x) = x^3 + 1$ has only one real zero. The following theorem, called **Descartes's Rule of Signs,** sheds more light on the number of real zeros of a polynomial.

Descartes's Rule of Signs

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial with real coefficients and $a_0 \neq 0$.

1. The number of *positive real zeros* of f is either equal to the number of variations in sign of $f(x)$ or less than that number by an even integer.
2. The number of *negative real zeros* of f is either equal to the number of variations in sign of $f(-x)$ or less than that number by an even integer.

A **variation in sign** means that two consecutive coefficients have opposite signs.

When using Descartes's Rule of Signs, a zero of multiplicity k should be counted as k zeros. For instance, the polynomial $x^3 - 3x + 2$ has two variations in sign, and so has either two positive or no positive real zeros. Because

$$x^3 - 3x + 2 = (x - 1)(x - 1)(x + 2)$$

you can see that the two positive real zeros are $x = 1$ of multiplicity 2.

Example 9 ▶ Using Descartes's Rule of Signs

Describe the possible real zeros of

$$f(x) = 3x^3 - 5x^2 + 6x - 4.$$

Solution

The original polynomial has *three* variations in sign.

+ to −   + to −

$$f(x) = 3x^3 - 5x^2 + 6x - 4$$

− to +

The polynomial

$$f(-x) = 3(-x)^3 - 5(-x)^2 + 6(-x) - 4$$
$$= -3x^3 - 5x^2 - 6x - 4$$

has no variations in sign. So, from Descartes's Rule of Signs, the polynomial $f(x) = 3x^3 - 5x^2 + 6x - 4$ has either three positive real zeros or one positive real zero, and has no negative real zeros. From the graph in Figure 3.33, you can see that the function has only one real zero (it is a positive number, near $x = 1$).

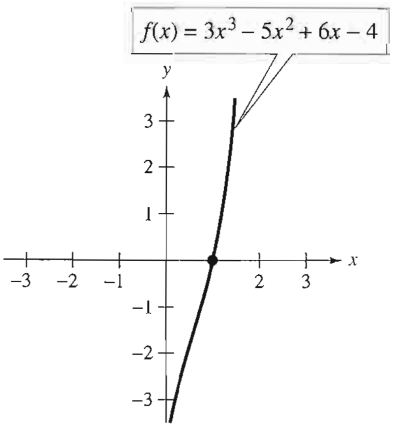


FIGURE 3.33

Another test for zeros of a polynomial function is related to the sign pattern in the last row of the synthetic division array. This test can give you an upper or lower bound of the real zeros of f. A real number b is an **upper bound** for the real zeros of f if no zeros are greater than b. Similarly, b is a **lower bound** if no real zeros of f are less than b.

Upper and Lower Bound Rules

Let $f(x)$ be a polynomial with real coefficients and a positive leading coefficient. Suppose $f(x)$ is divided by $x - c$, using synthetic division.

1. If $c > 0$ and each number in the last row is either positive or zero, c is an *upper bound* for the real zeros of f.
2. If $c < 0$ and the numbers in the last row are alternately positive and negative (zero entries count as positive or negative), c is a *lower bound* for the real zeros of f.

Example 10 ▶ Finding the Zeros of a Polynomial Function

Find the real zeros of

$$f(x) = 6x^3 - 4x^2 + 3x - 2.$$

Solution

The possible real zeros are as follows.

$$\frac{\text{Factors of } 2}{\text{Factors of } 6} = \frac{\pm1, \pm2}{\pm1, \pm2, \pm3, \pm6}$$

$$= \pm1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm\frac{2}{3}, \pm2$$

Because $f(x)$ has three variations in sign and $f(-x)$ has none, you can apply Descartes's Rule of Signs to conclude that there are three positive real zeros or one positive real zero, and no negative zeros. Trying $x = 1$ produces the following.

$$\begin{array}{r|rrrr} 1 & 6 & -4 & 3 & -2 \\ & & 6 & 2 & 5 \\ \hline & 6 & 2 & 5 & 3 \end{array}$$

So, $x = 1$ is not a zero, but because the last row has all positive entries, you know that $x = 1$ is an upper bound for the real zeros. So, you can restrict the search to zeros between 0 and 1. By trial and error, you can determine that $x = \frac{2}{3}$ is a zero. So,

$$f(x) = \left(x - \frac{2}{3}\right)(6x^2 + 3).$$

Because $6x^2 + 3$ has no real zeros, it follows that $x = \frac{2}{3}$ is the only real zero.

Before concluding this section, here are two additional hints that can help you find the real zeros of a polynomial.

1. If the terms of $f(x)$ have a common monomial factor, it should be factored out before applying the tests in this section. For instance, by writing

$$f(x) = x^4 - 5x^3 + 3x^2 + x = x(x^3 - 5x^2 + 3x + 1)$$

you can see that $x = 0$ is a zero of f and that the remaining zeros can be obtained by analyzing the cubic factor.

2. If you are able to find all but two zeros of $f(x)$, you can always use the Quadratic Formula on the remaining quadratic factor. For instance, if you succeeded in writing

$$f(x) = x^4 - 5x^3 + 3x^2 + x = x(x - 1)(x^2 - 4x - 1)$$

you can apply the Quadratic Formula to $x^2 - 4x - 1$ to conclude that the two remaining zeros are $x = 2 + \sqrt{5}$ and $x = 2 - \sqrt{5}$.

Example 11 ▶ Using a Polynomial Model

You are designing candle-making kits. Each kit contains 25 cubic inches of candle wax and a mold for making a pyramid-shaped candle. You want the height of the candle to be 2 inches less than the length of each side of the candle's square base. What should the dimensions of your candle mold be?

Solution

The volume of a pyramid is $V = \frac{1}{3}Bh$, where B is the area of the base and h is the height. The area of the base is x^2 and the height is $(x - 2)$. So, the volume of the pyramid is $V = \frac{1}{3}x^2(x - 2)$. Substituting 25 for the volume yields the following.

$$25 = \frac{1}{3}x^2(x - 2) \qquad \text{Substitute 25 for } V.$$

$$75 = x^3 - 2x^2 \qquad \text{Multiply each side by 3.}$$

$$0 = x^3 - 2x^2 - 75 \qquad \text{Write in general form.}$$

The possible rational zeros are $x = \pm1, \pm3, \pm5, \pm15, \pm25, \pm75$. Using synthetic division, you can determine that $x = 5$ is a solution. The other two solutions, which satisfy $x^2 + 3x + 15 = 0$, are imaginary and can be discarded. You can conclude that the base of the candle mold should be 5 inches by 5 inches and the height of the mold should be $5 - 2 = 3$ inches.

Writing ABOUT MATHEMATICS

Factoring Polynomials Compile a list of all the various techniques for factoring a polynomial that have been covered so far in the text. Give an example illustrating each technique, and write a paragraph discussing when the use of each technique is appropriate.

3.4 Exercises

In Exercises 1–6, find all the zeros of the function.

1. $f(x) = x(x - 6)^2$

2. $f(x) = x^2(x + 3)(x^2 - 1)$

3. $g(x) = (x - 2)(x + 4)^3$

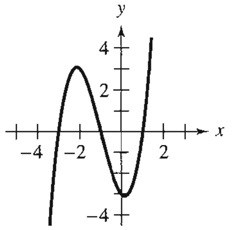
4. $f(x) = (x + 5)(x - 8)^2$

5. $f(x) = (x + 6)(x + i)(x - i)$

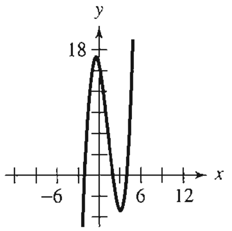
6. $h(t) = (t - 3)(t - 2)(t - 3i)(t + 3i)$

In Exercises 7–10, use the Rational Zero Test to list all possible rational zeros of *f*. Verify that the zeros of *f* shown on the graph are contained in the list.

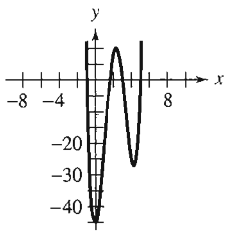
7. $f(x) = x^3 + 3x^2 - x - 3$



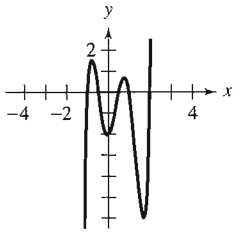
8. $f(x) = x^3 - 4x^2 - 4x + 16$



9. $f(x) = 2x^4 - 17x^3 + 35x^2 + 9x - 45$



10. $f(x) = 4x^5 - 8x^4 - 5x^3 + 10x^2 + x - 2$



In Exercises 11–20, find all the real zeros of the function.

11. $f(x) = x^3 - 6x^2 + 11x - 6$

12. $f(x) = x^3 - 7x - 6$

13. $g(x) = x^3 - 4x^2 - x + 4$

14. $h(x) = x^3 - 9x^2 + 20x - 12$

15. $h(t) = t^3 + 12t^2 + 21t + 10$

16. $p(x) = x^3 - 9x^2 + 27x - 27$

17. $C(x) = 2x^3 + 3x^2 - 1$

18. $f(x) = 3x^3 - 19x^2 + 33x - 9$

19. $f(x) = 9x^4 - 9x^3 - 58x^2 + 4x + 24$

20. $f(x) = 2x^4 - 15x^3 + 23x^2 + 15x - 25$

In Exercises 21–24, find all real solutions of the polynomial equation.

21. $z^4 - z^3 - 2z - 4 = 0$

22. $x^4 - 13x^2 - 12x = 0$

23. $2y^4 + 7y^3 - 26y^2 + 23y - 6 = 0$

24. $x^5 - x^4 - 3x^3 + 5x^2 - 2x = 0$

In Exercises 25–28, (a) list the possible rational zeros of *f*, (b) sketch the graph of *f* so that some of the possible zeros in part (a) can be disregarded, and then (c) determine all real zeros of *f*.

25. $f(x) = x^3 + x^2 - 4x - 4$

26. $f(x) = -3x^3 + 20x^2 - 36x + 16$

27. $f(x) = -4x^3 + 15x^2 - 8x - 3$

28. $f(x) = 4x^3 - 12x^2 - x + 15$

In Exercises 29–32, (a) list the possible rational zeros of *f*, (b) use a graphing utility to graph *f* so that some of the possible zeros in part (a) can be disregarded, and then (c) determine all real zeros of *f*.

29. $f(x) = -2x^4 + 13x^3 - 21x^2 + 2x + 8$

30. $f(x) = 4x^4 - 17x^2 + 4$

31. $f(x) = 32x^3 - 52x^2 + 17x + 3$

32. $f(x) = 4x^3 + 7x^2 - 11x - 18$

Graphical Analysis In Exercises 33–36, (a) use the *zero* or *root* feature of a graphing utility to approximate the zeros of the function accurate to three decimal places and (b) determine one of the exact zeros, use synthetic division to verify your result, and then factor the polynomial completely.

33. $f(x) = x^4 - 3x^2 + 2$

34. $P(t) = t^4 - 7t^2 + 12$

35. $h(x) = x^5 - 7x^4 + 10x^3 + 14x^2 - 24x$

36. $g(x) = 6x^4 - 11x^3 - 51x^2 + 99x - 27$

In Exercises 37–42, find a polynomial function with integer coefficients that has the given zeros. (There are many correct answers.)

37. $1, 5i, -5i$

38. $4, 3i, -3i$

39. $6, -5 + 2i, -5 - 2i$

40. $2, 4 + i, 4 - i$

41. $\frac{2}{3}, -1, 3 + \sqrt{2}i$

42. $-5, -5, 1 + \sqrt{3}i$

In Exercises 43–46, write the polynomial (a) as the product of factors that are irreducible over the *rationals*, (b) as the product of linear and quadratic factors that are irreducible over the *reals*, and (c) in completely factored form.

43. $f(x) = x^4 + 6x^2 - 27$

44. $f(x) = x^4 - 2x^3 - 3x^2 + 12x - 18$
(*Hint:* One factor is $x^2 - 6$.)

45. $f(x) = x^4 - 4x^3 + 5x^2 - 2x - 6$
(*Hint:* One factor is $x^2 - 2x - 2$.)

46. $f(x) = x^4 - 3x^3 - x^2 - 12x - 20$
(*Hint:* One factor is $x^2 + 4$.)

In Exercises 47–54, use the given zero to find all the zeros of the function.

	Function	*Zero*
47.	$f(x) = 2x^3 + 3x^2 + 50x + 75$	$5i$
48.	$f(x) = x^3 + x^2 + 9x + 9$	$3i$
49.	$f(x) = 2x^4 - x^3 + 7x^2 - 4x - 4$	$2i$
50.	$g(x) = x^3 - 7x^2 - x + 87$	$5 + 2i$
51.	$g(x) = 4x^3 + 23x^2 + 34x - 10$	$-3 + i$
52.	$h(x) = 3x^3 - 4x^2 + 8x + 8$	$1 - \sqrt{3}i$
53.	$f(x) = x^4 + 3x^3 - 5x^2 - 21x + 22$	$-3 + \sqrt{2}i$
54.	$f(x) = x^3 + 4x^2 + 14x + 20$	$-1 - 3i$

In Exercises 55–72, find all the zeros of the function and write the polynomial as a product of linear factors.

55. $f(x) = x^2 + 25$

56. $f(x) = x^2 - x + 56$

57. $h(x) = x^2 - 4x + 1$

58. $g(x) = x^2 + 10x + 23$

59. $f(x) = x^4 - 81$

60. $f(y) = y^4 - 625$

61. $f(z) = z^2 - 2z + 2$

62. $h(x) = x^3 - 3x^2 + 4x - 2$

63. $g(x) = x^3 - 6x^2 + 13x - 10$

64. $f(x) = x^3 - 2x^2 - 11x + 52$

65. $h(x) = x^3 - x + 6$

66. $h(x) = x^3 + 9x^2 + 27x + 35$

67. $f(x) = 5x^3 - 9x^2 + 28x + 6$

68. $g(x) = 3x^3 - 4x^2 + 8x + 8$

69. $g(x) = x^4 - 4x^3 + 8x^2 - 16x + 16$

70. $h(x) = x^4 + 6x^3 + 10x^2 + 6x + 9$

71. $f(x) = x^4 + 10x^2 + 9$

72. $f(x) = x^4 + 29x^2 + 100$

In Exercises 73–78, find all the zeros of the function. When there is an extended list of possible rational zeros, use a graphing utility to graph the function in order to discard any rational zeros that are obviously not zeros of the function.

73. $f(x) = x^3 + 24x^2 + 214x + 740$

74. $f(s) = 2s^3 - 5s^2 + 12s - 5$

75. $f(x) = 16x^3 - 20x^2 - 4x + 15$

76. $f(x) = 9x^3 - 15x^2 + 11x - 5$

77. $f(x) = 2x^4 + 5x^3 + 4x^2 + 5x + 2$

78. $g(x) = x^5 - 8x^4 + 28x^3 - 56x^2 + 64x - 32$

In Exercises 79–86, use Descartes's Rule of Signs to determine the possible number of positive and negative zeros of the function.

79. $g(x) = 5x^5 + 10x$

80. $h(x) = 4x^2 - 8x + 3$

81. $h(x) = 3x^4 + 2x^2 + 1$

82. $h(x) = 2x^4 - 3x + 2$

83. $g(x) = 2x^3 - 3x^2 - 3$

84. $f(x) = 4x^3 - 3x^2 + 2x - 1$

85. $f(x) = -5x^3 + x^2 - x + 5$

86. $f(x) = 3x^3 + 2x^2 + x + 3$

In Exercises 87–90, use synthetic division to verify the upper and lower bounds of the real zeros of *f*.

87. $f(x) = x^4 - 4x^3 + 15$

(a) Upper: $x = 4$ (b) Lower: $x = -1$

88. $f(x) = 2x^3 - 3x^2 - 12x + 8$

(a) Upper: $x = 4$ (b) Lower: $x = -3$

89. $f(x) = x^4 - 4x^3 + 16x - 16$

(a) Upper: $x = 5$ (b) Lower: $x = -3$

90. $f(x) = 2x^4 - 8x + 3$

(a) Upper: $x = 3$ (b) Lower: $x = -4$

In Exercises 91–94, find all the real zeros of the function.

91. $f(x) = 4x^3 - 3x - 1$

92. $f(z) = 12z^3 - 4z^2 - 27z + 9$

93. $f(y) = 4y^3 + 3y^2 + 8y + 6$

94. $g(x) = 3x^3 - 2x^2 + 15x - 10$

In Exercises 95–98, find all the rational zeros of the polynomial function.

95. $P(x) = x^4 - \frac{25}{4}x^2 + 9 = \frac{1}{4}(4x^4 - 25x^2 + 36)$

96. $f(x) = x^3 - \frac{3}{2}x^2 - \frac{23}{2}x + 6 = \frac{1}{2}(2x^3 - 3x^2 - 23x + 12)$

97. $f(x) = x^3 - \frac{1}{4}x^2 - x + \frac{1}{4} = \frac{1}{4}(4x^3 - x^2 - 4x + 1)$

98. $f(z) = z^3 + \frac{11}{6}z^2 - \frac{1}{2}z - \frac{1}{3} = \frac{1}{6}(6z^3 + 11z^2 - 3z - 2)$

In Exercises 99–102, match the cubic function with the numbers of rational and irrational zeros.

(a) Rational zeros: 0; Irrational zeros: 1

(b) Rational zeros: 3; Irrational zeros: 0

(c) Rational zeros: 1; Irrational zeros: 2

(d) Rational zeros: 1; Irrational zeros: 0

99. $f(x) = x^3 - 1$

100. $f(x) = x^3 - 2$

101. $f(x) = x^3 - x$

102. $f(x) = x^3 - 2x$

103. ***Geometry*** An open box is to be made from a rectangular piece of material, 15 centimeters by 9 centimeters, by cutting equal squares from the corners and turning up the sides.

(a) Let x represent the length of the sides of the squares removed. Draw a diagram showing the squares removed from the original piece of material and the resulting dimensions of the open box.

(b) Use the diagram to write the volume V of the box as a function of x. Determine the domain of the function.

(c) Sketch the graph of the function and approximate the dimensions of the box that will yield a maximum volume.

(d) Find values of x such that $V = 56$. Which of these values is a physical impossibility in the construction of the box? Explain.

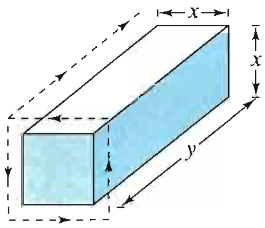
104. ***Geometry*** A rectangular package to be sent by a delivery service (see figure) can have a maximum combined length and girth (perimeter of a cross section) of 120 inches.

(a) Show that the volume of the package is

$$V(x) = 4x^2(30 - x).$$

(b) Use a graphing utility to graph the function and approximate the dimensions of the package that will yield a maximum volume.

(c) Find values of x such that $V = 13{,}500$. Which of these values is a physical impossibility in the construction of the package? Explain.



105. ***Advertising Cost*** A company that produces portable cassette players estimates that the profit P (in dollars) for selling a particular model is

$$P = -76x^3 + 4830x^2 - 320{,}000, \quad 0 \le x \le 60$$

where x is the advertising expense (in tens of thousands of dollars). Using this model, find the smaller of two advertising amounts that will yield a profit of \$2,500,000.

106. ***Advertising Cost*** A company that manufactures bicycles estimates that the profit P (in dollars) for selling a particular model is

$$P = -45x^3 + 2500x^2 - 275{,}000, \quad 0 \le x \le 50$$

where x is the advertising expense (in tens of thousands of dollars). Using this model, find the smaller of two advertising amounts that will yield a profit of \$800,000.

▶ Model It

107. ***Athletics*** The attendance A (in millions) at NCAA women's college basketball games for the years 1994 through 2000 is shown in the table, where t represents the year, with $t = 4$ corresponding to 1994. (Source: National Collegiate Athletic Association)

Year, t	Attendance, A
4	4.557
5	4.962
6	5.234
7	6.734
8	7.387
9	8.698
10	8.825

(a) Use the *regression* feature of a graphing utility to find a cubic model for the data.

(b) Use the graphing utility to create a scatter plot of the data. Then graph the model and the scatter plot in the same viewing window. How do they compare?

(c) According to the model found in part (a), in what year did attendance reach 5.5 million?

(d) According to the model found in part (a), in what year did attendance reach 8 million?

(e) According to the right-hand behavior of the model, will the attendance continue to increase? Explain.

108. ***Cost*** The ordering and transportation cost C (in thousands of dollars) for the components used in manufacturing a product is

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x+30}\right), \qquad x \geq 1$$

where x is the order size (in hundreds). In calculus, it can be shown that the cost is a minimum when

$$3x^3 - 40x^2 - 2400x - 36{,}000 = 0.$$

Use a calculator to approximate the optimal order size to the nearest hundred units.

109. ***Height of a Baseball*** A baseball is thrown upward from ground level with an initial velocity of 48 feet per second, and its height h (in feet) is

$$h(t) = -16t^2 + 48t, \qquad 0 \leq t \leq 3$$

where t is the time (in seconds). You are told the ball reaches a height of 64 feet. Is this possible?

110. ***Profit*** The demand equation for a certain product is $p = 140 - 0.0001x$, where p is the unit price (in dollars) of the product and x is the number of units produced and sold. The cost equation for the product is $C = 80x + 150{,}000$, where C is the total cost (in dollars) and x is the number of units produced. The total profit obtained by producing and selling x units is

$$P = R - C = xp - C.$$

You are working in the marketing department of the company that produces this product, and you are asked to determine a price p that will yield a profit of 9 million dollars. Is this possible? Explain.

Synthesis

True or False? **In Exercises 111 and 112, decide whether the statement is true or false. Justify your answer.**

111. It is possible for a third-degree polynomial function with integer coefficients to have no real zeros.

112. If $x = -i$ is a zero of the function $f(x) = x^3 + ix^2 + ix - 1$, then $x = i$ must also be a zero of f.

Think About It **In Exercises 113–118, determine (if possible) the zeros of the function g if the function f has zeros at $x = r_1$, $x = r_2$, and $x = r_3$.**

113. $g(x) = -f(x)$

114. $g(x) = 3f(x)$

115. $g(x) = f(x - 5)$

116. $g(x) = f(2x)$

117. $g(x) = 3 + f(x)$

118. $g(x) = f(-x)$

119. ***Exploration*** Use a graphing utility to graph the function $f(x) = x^4 - 4x^2 + k$ for different values of k. Find values of k such that the zeros of f satisfy the specified characteristics. (Some parts do not have unique answers.)

(a) Four real zeros

(b) Two real zeros, each of multiplicity 2

(c) Two real zeros and two complex roots

(d) Four complex zeros

120. ***Think About It*** Will the answers to Exercise 119 change for the function g?

(a) $g(x) = f(x - 2)$ (b) $g(x) = f(2x)$

121. ***Think About It*** A third-degree polynomial function f has real zeros -2, $\frac{1}{2}$, and 3, and its leading coefficient is negative. Write an equation for f. Sketch the graph of f. How many different polynomial functions are possible for f?

122. ***Think About It*** Sketch the graph of a fifth-degree polynomial function whose leading coefficient is positive and that has one root at $x = 3$ of multiplicity 2.

123. Use the information in the table.

Interval	Value of $f(x)$
$(-\infty, -2)$	Positive
$(-2, 1)$	Negative
$(1, 4)$	Negative
$(4, \infty)$	Positive

(a) What are the three real zeros of the polynomial function f?

(b) What can be said about the behavior of the graph of f at $x = 1$?

(c) What is the least possible degree of f? Explain. Can the degree of f ever be odd? Explain.

(d) Is the leading coefficient of f positive or negative? Explain.

(e) Write an equation for f. There are many correct answers.

(f) Sketch a graph of the equation you wrote in part (e).

124. Use the information in the table.

Interval	Value of $f(x)$
$(-\infty, -2)$	Negative
$(-2, 0)$	Positive
$(0, 2)$	Positive
$(2, \infty)$	Negative

(a) What are the three real zeros of the polynomial function f?

(b) What can be said about the behavior of the graph of f at $x = 0$?

(c) What is the least possible degree of f? Explain. Can the degree of f ever be odd? Explain.

(d) Is the leading coefficient of f positive or negative? Explain.

(e) Write an equation for f. There are many correct answers.

(f) Sketch a graph of the equation you wrote in part (e).

125. (a) Find a quadratic function f (with integer coefficients) that has $\pm\sqrt{b}\,i$ as zeros. Assume that b is a positive integer.

(b) Find a quadratic function f (with integer coefficients) that has $a \pm bi$ as zeros. Assume that b is a positive integer.

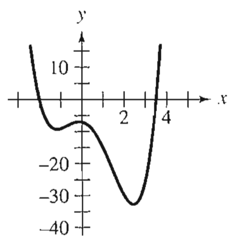
126. ***Graphical Reasoning*** The graph of one of the following functions is shown below. Identify the function shown in the graph. Explain why each of the others is not the correct function. Use a graphing utility to verify your result.

(a) $f(x) = x^2(x + 2)(x - 3.5)$

(b) $g(x) = (x + 2)(x - 3.5)$

(c) $h(x) = (x + 2)(x - 3.5)(x^2 + 1)$

(d) $k(x) = (x + 1)(x + 2)(x - 3.5)$



Review

In Exercises 127–130, perform the operation and simplify.

127. $(-3 + 6i) - (8 - 3i)$ **128.** $(12 - 5i) + 16i$

129. $(6 - 2i)(1 + 7i)$ **130.** $(9 - 5i)(9 + 5i)$

In Exercises 131–136, use the graph of f to sketch the graph of g. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

131. $g(x) = f(x - 2)$

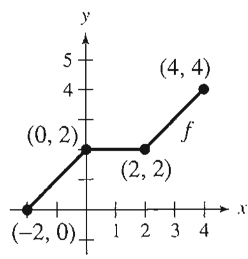
132. $g(x) = f(x) - 2$

133. $g(x) = 2f(x)$

134. $g(x) = f(-x)$

135. $g(x) = f(2x)$

136. $g(x) = f\left(\frac{1}{2}x\right)$



3.5 Mathematical Modeling

▶ What you should learn

- How to use mathematical models to approximate sets of data points
- How to write mathematical models for direct variation
- How to write mathematical models for direct variation as an *n*th power
- How to write mathematical models for inverse variation
- How to write mathematical models for joint variation
- How to use the *regression* feature of a graphing utility to find the equation of a least squares regression line

▶ Why you should learn it

You can use functions as models to represent a wide variety of real-life data sets. For instance, in Exercise 63 on page 317, a variation model can be used to model the water temperature of the ocean at various depths.

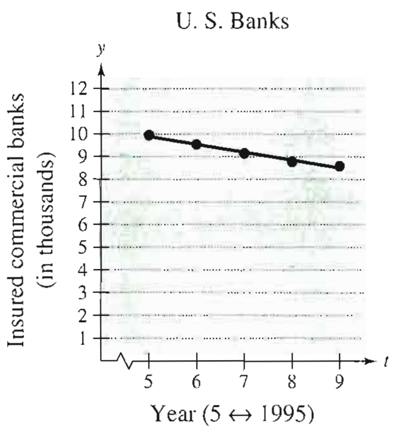


FIGURE 3.34

Introduction

You have already studied some techniques for fitting models to data. For instance, in Section 2.1, you learned how to find the equation of a line that passes through two points. In this section, you will study other techniques for fitting models to data: *direct and inverse variation* and *least squares regression*. The resulting models are either polynomial functions or rational functions. (Rational functions will be studied in Chapter 4.)

Example 1 ▶ A Mathematical Model

The numbers of insured commercial banks y (in thousands) in the United States for the years 1995 to 1999 are shown in the table. (Source: Federal Deposit Insurance Corporation)

Year	Insured commercial banks, y
1995	9.94
1996	9.53
1997	9.14
1998	8.77
1999	8.58

A linear model that approximates this data is $y = -0.348t + 11.63$ for $5 \le t \le 9$, where t is the year, with $t = 5$ corresponding to 1995. Plot the actual data *and* the model on the same graph. How closely does the model represent the data?

Solution

The actual data is plotted in Figure 3.34, along with the graph of the linear model. From the graph, it appears that the model is a "good fit" for the actual data. You can see how well the model fits by comparing the actual values of y with the values of y given by the model. The values given by the model are labeled y^* in the table below.

t	5	6	7	8	9
y	9.94	9.53	9.14	8.77	8.58
y^*	9.89	9.54	9.19	8.85	8.50

Note in Example 1 that you could have chosen any two points to find a line that fits the data. However, the given linear model was found using the *regression* feature of a graphing utility and is the line that *best* fits the data. This concept of a "best-fitting" line is discussed later in this section.

Direct Variation

There are two basic types of linear models. The more general model has a y-intercept that is nonzero.

$$y = mx + b, \quad b \neq 0$$

The simpler model

$$y = kx$$

has a y-intercept that is zero. In the simpler model, y is said to **vary directly** as x, or to be **directly proportional** to x.

Direct Variation

The following statements are equivalent.

1. y **varies directly** as x.
2. y is **directly proportional** to x.
3. $y = kx$ for some nonzero constant k.

k is the **constant of variation** or the **constant of proportionality.**

Example 2 ▶ Direct Variation

In Pennsylvania, the state income tax is directly proportional to *gross income*. You are working in Pennsylvania and your state income tax deduction is \$42 for a gross monthly income of \$1500. Find a mathematical model that gives the Pennsylvania state income tax in terms of gross income.

Solution

Verbal Model: State income tax $=$ k $\cdot$ Gross income

Labels: State income tax $= y$ (dollars)
Gross income $= x$ (dollars)
Income tax rate $= k$ (percent in decimal form)

Equation: $y = kx$

To solve for k, substitute the given information into the equation $y = kx$, and then solve for k.

$y = kx$	Write direct variation model.
$42 = k(1500)$	Substitute $y = 42$ and $x = 1500$.
$0.028 = k$	Simplify.

So, the equation (or model) for state income tax in Pennsylvania is

$$y = 0.028x.$$

In other words, Pennsylvania has a state income tax rate of 2.8% of gross income. The graph of this equation is shown in Figure 3.35.

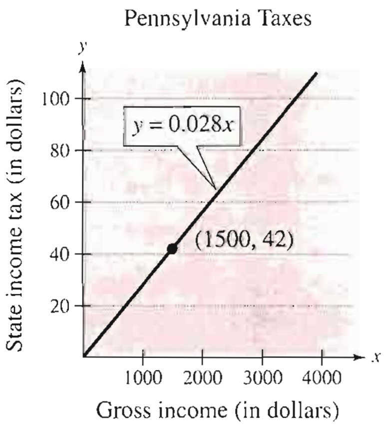


FIGURE 3.35

Direct Variation as an *n*th Power

Another type of direct variation relates one variable to a *power* of another variable. For example, in the formula for the area of a circle

$$A = \pi r^2$$

the area A is directly proportional to the square of the radius r. Note that for this formula, π is the constant of proportionality.

STUDY TIP

Note that the direct variation model $y = kx$ is a special case of $y = kx^n$ with $n = 1$.

Direct Variation as an *n*th Power

The following statements are equivalent.

1. y **varies directly as the *n*th power** of x.
2. y is **directly proportional to the *n*th power** of x.
3. $y = kx^n$ for some constant k.

Example 3 ▶ Direct Variation as *n*th Power

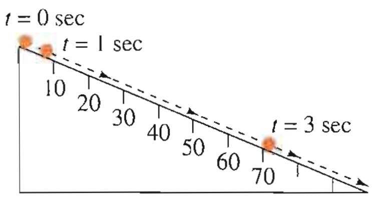


FIGURE 3.36

The distance a ball rolls down an inclined plane is directly proportional to the square of the time it rolls. During the first second, the ball rolls 8 feet. (See Figure 3.36.)

a. Write an equation relating the distance traveled to the time.

b. How far will the ball roll during the first 3 seconds?

Solution

a. Letting d be the distance (in feet) the ball rolls and letting t be the time (in seconds), you have

$$d = kt^2.$$

Now, because $d = 8$ when $t = 1$, you can see that $k = 8$, as follows.

$$d = kt^2$$

$$8 = k(1)^2$$

$$8 = k$$

So, the equation relating distance to time is

$$d = 8t^2.$$

b. When $t = 3$, the distance traveled is $d = 8(3)^2 = 8(9) = 72$ feet.

In Examples 2 and 3, the direct variations are such that an *increase* in one variable corresponds to an *increase* in the other variable. This is also true in the model $d = \frac{1}{5}F$, $F > 0$, where an increase in F results in an increase in d. You should not, however, assume that this always occurs with direct variation. For example, in the model $y = -3x$, an increase in x results in a *decrease* in y, and yet y is said to vary directly as x.

Inverse Variation

> **Inverse Variation**
>
> The following statements are equivalent.
>
> **1.** y **varies inversely** as x. **2.** y is **inversely proportional** to x.
>
> **3.** $y = \frac{k}{x}$ for some constant k.

If x and y are related by an equation of the form $y = k/x^n$, then y varies inversely as the nth power of x (or y is inversely proportional to the nth power of x).

Example 4 ▶ Inverse Variation

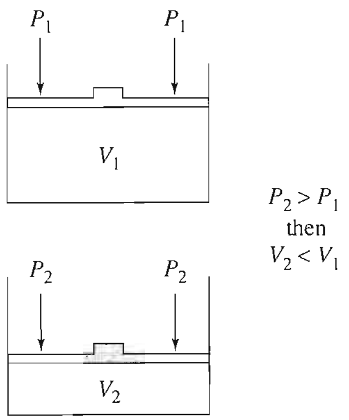


FIGURE 3.37 *If the temperature is held constant and pressure increases, volume decreases.*

A gas law states that the volume of an enclosed gas varies directly as the temperature *and* inversely as the pressure, as shown in Figure 3.37. The pressure of a gas is 0.75 kilogram per square centimeter when the temperature is 294 K and the volume is 8000 cubic centimeters.

a. Write an equation relating pressure, temperature, and volume.

b. Find the pressure when the temperature is 300 K and the volume is 7000 cubic centimeters.

Solution

a. Let V be volume (in cubic centimeters), let P be pressure (in kilograms per square centimeter), and let T be temperature (in Kelvin). Because V varies directly as T and inversely as P,

$$V = \frac{kT}{P}.$$

Now, because $P = 0.75$ when $T = 294$ and $V = 8000$,

$$8000 = \frac{k(294)}{0.75}$$

$$\frac{8000(0.75)}{294} = k$$

$$k = \frac{6000}{294} = \frac{1000}{49}.$$

So, the equation relating pressure, temperature, and volume is

$$V = \frac{1000}{49}\left(\frac{T}{P}\right).$$

b. When $T = 300$ and $V = 7000$, the pressure is

$$P = \frac{1000}{49}\left(\frac{300}{7000}\right) = \frac{300}{343} \approx 0.87 \text{ kilogram per square centimeter.}$$

Joint Variation

In Example 4, note that when a direct variation and an inverse variation occur in the same statement, they are coupled with the word "and." To describe two different *direct* variations in the same statement, the word **jointly** is used.

Joint Variation

The following statements are equivalent.

1. z **varies jointly** as x and y.

2. z is **jointly proportional** to x and y.

3. $z = kxy$ for some constant k.

If x, y, and z are related by an equation of the form

$$z = kx^n y^m$$

then z varies jointly as the nth power of x and the mth power of y.

Example 5 ▶ Joint Variation

The *simple* interest for a certain savings account is jointly proportional to the time and the principal. After one quarter (3 months), the interest on a principal of \$5000 is \$43.75.

a. Write an equation relating the interest, principal, and time.

b. Find the interest after three quarters.

Solution

a. Let I = interest (in dollars), P = principal (in dollars), and t = time (in years). Because I is jointly proportional to P and t,

$$I = kPt.$$

For $I = 43.75$, $P = 5000$, and $t = \frac{1}{4}$,

$$43.75 = k(5000)\left(\frac{1}{4}\right)$$

which implies that $k = 4(43.75)/5000 = 0.035$. So, the equation relating interest, principal, and time is

$$I = 0.035Pt$$

which is the familiar equation for simple interest where the constant of proportionality, 0.035, represents an annual interest rate of 3.5%.

b. When $P = \$5000$ and $t = \frac{3}{4}$, the interest is

$$I = (0.035)(5000)\left(\frac{3}{4}\right)$$

$$= \$131.25.$$

Least Squares Regression and Graphing Utilities

So far in this text, you have worked with many different types of mathematical models that approximate real-life data. In some instances the model was given, whereas in other instances you were asked to find the model using simple algebraic techniques or a graphing utility.

To find a model that approximates the data most accurately, statisticians use a measure called the **sum of square differences,** which is the sum of the squares of the differences between actual data values and model values. The "best-fitting" linear model is the one with the least sum of square differences. This best-fitting linear model is called the **least squares regression line.** Recall that you can approximate this line visually by plotting the data points and drawing the line that appears to fit best—or you can enter the data points into a calculator or computer and use the calculator's or computer's linear regression program. When you run a linear regression program, the "r-value" or **correlation coefficient** gives a measure of how well the model fits the data. The closer the value of $|r|$ is to 1, the better the fit.

Example 6 ▶ Finding a Least Squares Regression Line

The amounts p (in millions of dollars) of total annual prize money awarded at the Indianapolis 500 race from 1993 to 2001 are shown in the table. Construct a scatter plot that represents the data and find a linear model that approximates the data. (Source: Indy Racing League)

Year	Prize money, p
1993	7.68
1994	7.86
1995	8.06
1996	8.11
1997	8.61
1998	8.72
1999	9.05
2000	9.48
2001	9.62

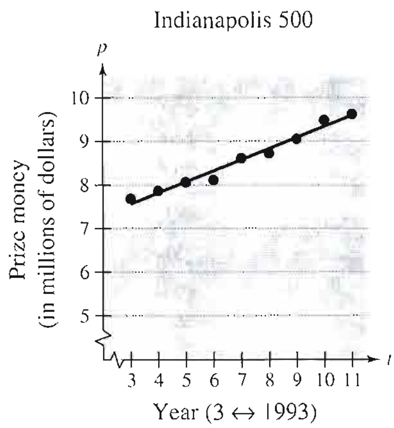


FIGURE 3.38

t	p	p^*
3	7.68	7.56
4	7.86	7.82
5	8.06	8.07
6	8.11	8.32
7	8.61	8.58
8	8.72	8.83
9	9.05	9.09
10	9.48	9.34
11	9.62	9.59

Solution

Let $t = 3$ represent 1993. The scatter plot for the points is shown in Figure 3.38. Using the *regression* feature of a graphing utility, you can determine that the equation of the least squares regression line is

$$p = 0.254t + 6.80.$$

To check this model, compare the actual p-values with the p-values given by the model, which are labeled p^* in the table at the left. The correlation coefficient for this model is $r \approx 0.988$, which implies that the model is a good fit.

3.5 Exercises

1. ***Employment*** The total numbers of employees (in thousands) in the United States from 1992 to 1999 are given by the following ordered pairs.

(1992, 128,105)	(1996, 133,943)
(1993, 129,200)	(1997, 136,297)
(1994, 131,056)	(1998, 137,673)
(1995, 132,304)	(1999, 139,368)

A linear model that approximates this data is $y = 124{,}420 + 1649.6t$, where y represents the number of employees (in thousands) and $t = 2$ represents 1992. Plot the actual data and the model on the same set of coordinate axes. How closely does the model represent the data? (Source: U.S. Bureau of Labor Statistics)

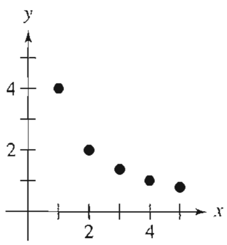
2. ***Sports*** The winning times (in minutes) in the women's 400-meter freestyle swimming event in the Olympics from 1948 to 2000 are given by the following ordered pairs.

(1948, 5.30)	(1976, 4.16)
(1952, 5.20)	(1980, 4.15)
(1956, 4.91)	(1984, 4.12)
(1960, 4.84)	(1988, 4.06)
(1964, 4.72)	(1992, 4.12)
(1968, 4.53)	(1996, 4.12)
(1972, 4.32)	(2000, 4.10)

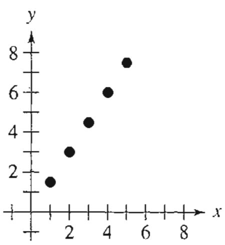
A linear model that approximates this data is $y = 5.06 - 0.024t$, where y represents the winning time in minutes and $t = 0$ represents 1950. Plot the actual data and the model on the same set of coordinate axes. How closely does the model represent the data? (Source: *The World Almanac and Book of Facts*)

Think About It **In Exercises 3 and 4, use the graph to determine whether y varies directly as some power of x or inversely as some power of x. Explain.**

3.



4.



In Exercises 5–8, use the given value of k to complete the table for the direct variation model $y = kx^2$. Plot the points on a rectangular coordinate system.

x	2	4	6	8	10
$y = kx^2$					

5. $k = 1$ **6.** $k = 2$

7. $k = \frac{1}{2}$ **8.** $k = \frac{1}{4}$

In Exercises 9–12, use the given value of k to complete the table for the inverse variation model

$$y = \frac{k}{x^2}.$$

Plot the points on a rectangular coordinate system.

x	2	4	6	8	10
$y = \dfrac{k}{x^2}$					

9. $k = 2$ **10.** $k = 5$

11. $k = 10$ **12.** $k = 20$

In Exercises 13–16, determine whether the variation model is of the form $y = kx$ or $y = k/x$, and find k.

13.

x	y
5	1
10	$\frac{1}{2}$
15	$\frac{1}{3}$
20	$\frac{1}{4}$
25	$\frac{1}{5}$

14.

x	y
5	2
10	4
15	6
20	8
25	10

15.

x	y
5	-3.5
10	-7
15	-10.5
20	-14
25	-17.5

16.

x	y
5	24
10	12
15	8
20	6
25	$\frac{24}{5}$

Direct Variation **In Exercises 17–20, assume that y is directly proportional to x. Use the given x-value and y-value to find a linear model that relates y and x.**

	x-Value	*y-Value*		*x-Value*	*y-Value*
17.	$x = 5$	$y = 12$	**18.**	$x = 2$	$y = 14$
19.	$x = 10$	$y = 2050$	**20.**	$x = 6$	$y = 580$

21. ***Simple Interest*** The simple interest on an investment is directly proportional to the amount of the investment. By investing \$2500 in a certain bond issue, you obtained an interest payment of \$87.50 after 1 year. Find a mathematical model that gives the interest I for this bond issue after 1 year in terms of the amount invested P.

22. ***Simple Interest*** The simple interest on an investment is directly proportional to the amount of the investment. By investing \$5000 in a municipal bond, you obtained an interest payment of \$187.50 after 1 year. Find a mathematical model that gives the interest I for this municipal bond after 1 year in terms of the amount invested P.

23. ***Measurement*** On a yardstick with scales in inches and centimeters, you notice that 13 inches is approximately the same length as 33 centimeters. Use this information to find a mathematical model that relates centimeters to inches. Then use the model to find the number of centimeters in 10 inches and 20 inches.

24. ***Measurement*** When buying gasoline, you notice that 14 gallons of gasoline is approximately the same amount of gasoline as 53 liters. Use this information to find a linear model that relates gallons to liters. Use the model to find the number of liters in 5 gallons and 25 gallons.

25. ***Taxes*** Property tax is based on the assessed value of the property. A house that has an assessed value of \$150,000 has a property tax of \$5520. Find a mathematical model that gives the amount of property tax y in terms of the assessed value x of the property. Use the model to find the property tax on a house that has an assessed value of \$200,000.

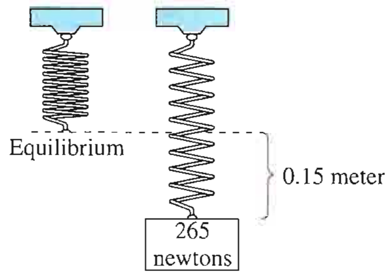
26. ***Taxes*** State sales tax is based on retail price. An item that sells for \$145.99 has a sales tax of \$10.22. Find a mathematical model that gives the amount of sales tax y in terms of the retail price x. Use the model to find the sales tax on a \$540.50 purchase.

Hooke's Law **In Exercises 27–30, use Hooke's Law for springs, which states that the distance a spring is stretched (or compressed) varies directly as the force on the spring.**

27. A force of 265 newtons stretches a spring 0.15 meter (see figure).

(a) How far will a force of 90 newtons stretch the spring?

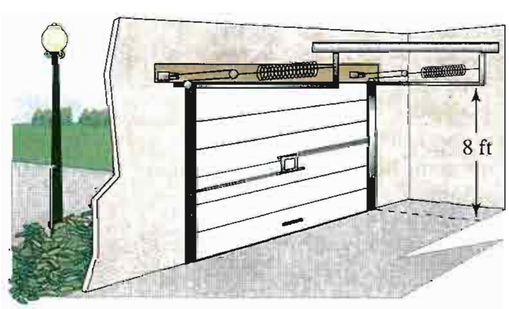
(b) What force is required to stretch the spring 0.1 meter?



28. A force of 220 newtons stretches a spring 0.12 meter. What force is required to stretch the spring 0.16 meter?

29. The coiled spring of a toy supports the weight of a child. The spring is compressed a distance of 1.9 inches by the weight of a 25-pound child. The toy will not work properly if its spring is compressed more than 3 inches. What is the weight of the heaviest child who should be allowed to use the toy?

30. An overhead garage door has two springs, one on each side of the door (see figure). A force of 15 pounds is required to stretch each spring 1 foot. Because of a pulley system, the springs stretch only one-half the distance the door travels. The door moves a total of 8 feet, and the springs are at their natural length when the door is open. Find the combined lifting force applied to the door by the springs when the door is closed.



In Exercises 31–40, find a mathematical model for the verbal statement.

31. A varies directly as the square of r.

32. V varies directly as the cube of e.

33. y varies inversely as the square of x.

34. h varies inversely as the square root of s.

35. F varies directly as g and inversely as r^2.

36. z is jointly proportional to the square of x and y^3.

37. Boyle's Law: For a constant temperature, the pressure P of a gas is inversely proportional to the volume V of the gas.

38. Newton's Law of Cooling: The rate of change R of the temperature of an object is proportional to the difference between the temperature T of the object and the temperature T_e of the environment in which the object is placed.

39. Newton's Law of Universal Gravitation: The gravitational attraction F between two objects of masses m_1 and m_2 is proportional to the product of the masses and inversely proportional to the square of the distance r between the objects.

40. Logistic growth: The rate of growth R of a population is jointly proportional to the size S of the population and the difference between S and the maximum population size L that the environment can support.

In Exercises 41–46, write a sentence using the variation terminology of this section to describe the formula.

41. Area of a triangle: $A = \frac{1}{2}bh$

42. Surface area of a sphere: $S = 4\pi r^2$

43. Volume of a sphere: $V = \frac{4}{3}\pi r^3$

44. Volume of a right circular cylinder: $V = \pi r^2 h$

45. Average speed: $r = \dfrac{d}{t}$

46. Free vibrations: $\omega = \sqrt{\dfrac{kg}{W}}$

In Exercises 47–54, find a mathematical model representing the statement. (In each case, determine the constant of proportionality.)

47. A varies directly as r^2. ($A = 9\pi$ when $r = 3$.)

48. y varies inversely as x. ($y = 3$ when $x = 25$.)

49. y is inversely proportional to x. ($y = 7$ when $x = 4$.)

50. z varies jointly as x and y. ($z = 64$ when $x = 4$ and $y = 8$.)

51. F is jointly proportional to r and the third power of s. ($F = 4158$ when $r = 11$ and $s = 3$.)

52. P varies directly as x and inversely as the square of y. $\left(P = \frac{28}{3}\text{ when } x = 42 \text{ and } y = 9.\right)$

53. z varies directly as the square of x and inversely as y. ($z = 6$ when $x = 6$ and $y = 4$.)

54. v varies jointly as p and q and inversely as the square of s. ($v = 1.5$ when $p = 4.1$, $q = 6.3$, and $s = 1.2$.)

Ecology **In Exercises 55 and 56, use the fact that the diameter of the largest particle that can be moved by a stream varies approximately directly as the square of the velocity of the stream.**

55. A stream with a velocity of $\frac{1}{4}$ mile per hour can move coarse sand particles about 0.02 inch in diameter. Approximate the velocity required to carry particles 0.12 inch in diameter.

56. A stream of velocity v can move particles of diameter d or less. By what factor does d increase when the velocity is doubled?

Resistance **In Exercises 57 and 58, use the fact that the resistance of a wire carrying an electrical current is directly proportional to its length and inversely proportional to its cross-sectional area.**

57. If #28 copper wire (which has a diameter of 0.0126 inch) has a resistance of 66.17 ohms per thousand feet, what length of #28 copper wire will produce a resistance of 33.5 ohms?

58. A 14-foot piece of copper wire produces a resistance of 0.05 ohm. Use the constant of proportionality from Exercise 57 to find the diameter of the wire.

59. ***Free Fall*** Neglecting air resistance, the distance s an object falls varies directly as the square of the duration t of the fall. An object falls a distance of 144 feet in 3 seconds. How far will it fall in 5 seconds?

60. ***Spending*** The prices of three sizes of pizza at a pizza shop are as follows.

9-inch: \$8.78, 12-inch: \$11.78, 15-inch: \$14.18

You would expect that the price of a certain size of pizza would be directly proportional to its surface area. Is that the case for this pizza shop? If not, which size of pizza is the best buy?

61. ***Fluid Flow*** The velocity v of a fluid flowing in a conduit is inversely proportional to the cross-sectional area of the conduit. (Assume that the volume of the flow per unit of time is held constant.) Determine the change in the velocity of water flowing from a hose when a person places a finger over the end of the hose to decrease its cross-sectional area by 25%.

62. ***Beam Load*** The maximum load that can be safely supported by a horizontal beam varies jointly as the width of the beam and the square of its depth, and inversely as the length of the beam. Determine the change in the maximum safe load under the following conditions.

(a) The width and length of the beam are doubled.

(b) The width and depth of the beam are doubled.

(c) All three of the dimensions are doubled.

(d) The depth of the beam is halved.

▶ Model It

63. ***Data Analysis*** An oceanographer took readings of the water temperature C (in degrees Celsius) at depth d (in meters). The data collected is shown in the table.

Depth, d	Temperature, C
1000	4.2°
2000	1.9°
3000	1.4°
4000	1.2°
5000	0.9°

(a) Sketch a scatter plot of the data.

(b) Does it appear that the data can be modeled by the inverse variation model $C = k/d$? If so, find k for each pair of coordinates.

(c) Determine the mean value of k from part (b) to find the inverse variation model $C = k/d$.

(d) Use a graphing utility to plot the data points and the inverse model in part (c).

(e) Use the model to approximate the depth at which the water temperature is 3°C.

64. ***Data Analysis*** An experiment in a physics lab requires a student to measure the compressed length y (in centimeters) of a spring when a force of F pounds is applied. The data is shown in the table.

Force, F	Length, y
0	0
2	1.15
4	2.3
6	3.45
8	4.6
10	5.75
12	6.9

(a) Sketch a scatter plot of the data.

(b) Does it appear that the data can be modeled by Hooke's Law? If so, estimate k. (See Exercises 27–30.)

(c) Use the model in part (b) to approximate the force required to compress the spring 9 centimeters.

65. ***Data Analysis*** A light probe is located x centimeters from a light source, and the intensity y (in microwatts per square centimeter) of the light is measured. The results are shown in the table.

x	y
30	0.1881
34	0.1543
38	0.1172
42	0.0998
46	0.0775
50	0.0645

A model for the data is $y = 262.76/x^{2.12}$.

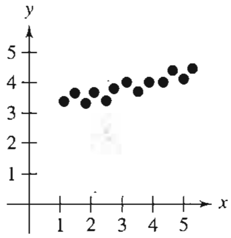
(a) Use a graphing utility to plot the data points and the model in the same viewing window.

(b) Use the model to approximate the light intensity 25 centimeters from the light source.

66. ***Illumination*** The illumination from a light source varies inversely as the square of the distance from the light source. When the distance from a light source is doubled, how does the illumination change? Discuss this model in terms of the data given in Exercise 65. Give a possible explanation of the difference.

In Exercises 67–70, sketch the line that you think best approximates the data in the scatter plot. Then find an equation of the line. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

67.



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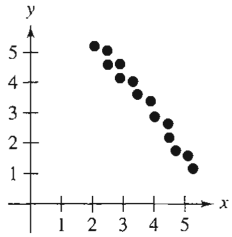

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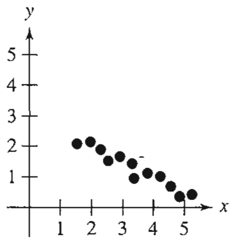

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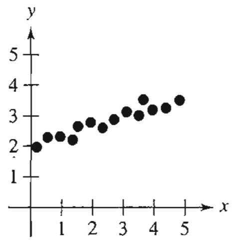


71. ***Sports*** The lengths (in feet) of the winning men's discus throws in the Olympics from 1908 to 2000 are listed below. (Source: *The World Almanac and Book of Facts*)

1908	134.2	1948	173.2	1976	221.4
1912	145.0	1952	180.5	1980	218.7
1920	146.6	1956	184.9	1984	218.5
1924	151.4	1960	194.2	1988	225.8
1928	155.2	1964	200.1	1992	213.7
1932	162.4	1968	212.5	1996	227.7
1936	165.6	1972	211.3	2000	227.3

(a) Sketch a scatter plot of the data. Let y represent the length of the winning discus throw (in feet) and let $t = 8$ represent 1908.

(b) Use a straightedge to sketch the best-fitting line through the points and find an equation of the line.

(c) Use the *regression* feature of a graphing utility to find the least squares regression line that fits this data.

(d) Compare the linear model you found in part (b) with the linear model given by the graphing utility in part (c).

(e) Use the models from parts (b) and (c) to estimate the winning men's discus throw in the year 2004.

(f) Use your school's library, the Internet, or some other reference source to analyze the accuracy of the estimate in part (e).

72. ***Sales*** The total sales (in millions of dollars) for Barnes & Noble from 1992 to 2000 are listed below. (Source: Barnes & Noble, Inc.)

1992	1086.7	1995	1976.9	1998	3005.6
1993	1337.4	1996	2448.1	1999	3486.0
1994	1622.7	1997	2796.8	2000	4375.8

(a) Sketch a scatter plot of the data. Let y represent the total sales (in millions of dollars) and let $t = 2$ represent 1992.

(b) Use a straightedge to sketch the best-fitting line through the points and find an equation of the line.

(c) Use the *regression* feature of a graphing utility to find the least squares regression line that fits this data.

(d) Compare the linear model you found in part (b) with the linear model given by the graphing utility in part (c).

(e) Use the models from parts (b) and (c) to estimate the sales of Barnes & Noble in 2002.

(f) Use your school's library, the Internet, or some other reference source to analyze the accuracy of the estimate in part (e).

73. ***Movie Theaters*** The table shows the annual receipts R (in millions of dollars) for motion picture movie theaters in the United States from 1993 through 2001. (Source: Motion Picture Association of America)

Year	Receipts, R
1993	5154
1994	5396
1995	5494
1996	5912
1997	6366
1998	6949
1999	7448
2000	7661
2001	8413

(a) Use a graphing utility to create a scatter plot of the data. Let $t = 3$ represent 1993.

(b) Use the *regression* feature of a graphing utility to find the equation of the least squares regression line that fits this data.

(c) Use the graphing utility to graph the scatter plot you found in part (a) and the model you found in part (b) in the same viewing window.

(d) Use the model to estimate the annual receipts in 2000 and 2002.

(e) Interpret the meaning of the slope of the linear model in the context of the problem.

74. ***Data Analysis*** The table shows the number x (in millions) of households with cable television and the number y (in millions) of daily newspapers in circulation in the United States from 1993 through 1999. (Source: Nielsen Media Research and Editor & Publisher Co.)

Households with cable, x	Daily newspapers, y
58.8	59.8
60.5	59.3
63.0	58.2
64.6	57.0
65.9	56.7
67.4	56.2
68.0	56.0

(a) Use the *regression* feature of a graphing utility to find the equation of the least squares regression line that fits this data.

(b) Use the graphing utility to create a scatter plot of the data. Then graph the model you found in part (a) and the scatter plot in the same viewing window.

(c) Use the model to estimate the number of daily newspapers in circulation if the number of households with cable television is 70 million.

(d) Interpret the meaning of the slope of the linear model in the context of the problem.

Synthesis

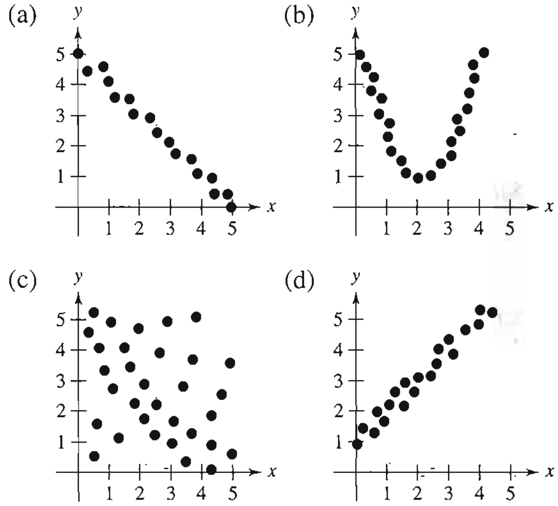
True or False? **In Exercises 75 and 76, decide whether the statement is true or false. Justify your answer.**

75. If y varies directly as x, then if x increases, y will increase as well.

76. In the equation for kinetic energy, $E = \frac{1}{2}mv^2$, the amount of kinetic energy E is directly proportional to the mass m of an object and the square of its velocity v.

77. ***Writing*** A linear mathematical model for predicting prize winnings at a race is based on data for 3 years. Write a paragraph discussing the potential accuracy or inaccuracy of such a model.

78. Discuss how well the data shown in each scatter plot can be approximated by a linear model.



Review

In Exercises 79–82, solve the inequality and graph the solution on the real number line.

79. $(x - 5)^2 \geq 1$

80. $3(x + 1)(x - 3) < 0$

81. $6x^3 - 30x^2 > 0$

82. $x^4(x - 8) \geq 0$

In Exercises 83 and 84, evaluate the function at each value of the independent variable and simplify.

83. $f(x) = \dfrac{x^2 + 5}{x - 3}$

(a) $f(0)$ (b) $f(-3)$ (c) $f(4)$

84. $f(x) = \begin{cases} -x^2 + 10, & x \geq -2 \\ 6x^2 - 1, & x < -2 \end{cases}$

(a) $f(-2)$ (b) $f(1)$ (c) $f(-8)$

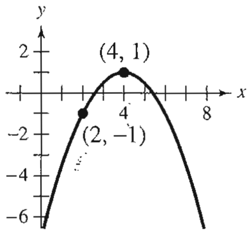
Chapter Summary

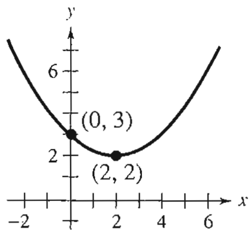
▶ *What* did you learn?

Section 3.1	Review Exercises
☐ How to analyze graphs of quadratic functions	1–6
☐ How to write quadratic functions in standard form and use the results to sketch graphs of functions	7–18
☐ How to use quadratic functions to model and solve real-life problems	19–24
Section 3.2	
☐ How to use transformations to sketch graphs of polynomial functions	25–30
☐ How to use the Leading Coefficient Test to determine the end behavior of graphs of polynomial functions	31–34
☐ How to use zeros of polynomial functions as sketching aids	35–44
☐ How to use the Intermediate Value Theorem to help locate zeros of polynomial functions	45–48
Section 3.3	
☐ How to use long division to divide polynomials by other polynomials	49–54
☐ How to use synthetic division to divide polynomials by binomials of the form $(x - k)$	55–62
☐ How to use the Remainder Theorem and the Factor Theorem	63–66
Section 3.4	
☐ How to use the Fundamental Theorem of Algebra to determine the number of zeros of polynomial functions	67–72
☐ How to find rational zeros of polynomial functions	73–80
☐ How to use conjugate pairs of complex zeros to find a polynomial with real coefficients	81, 82
☐ How to find zeros of polynomials by factoring	83–90
☐ How to use Descartes's Rule of Signs and the Upper and Lower Bound Rules to find zeros of polynomial functions	91–98
Section 3.5	
☐ How to use mathematical models to approximate sets of data points	99
☐ How to write mathematical models for direct variation	100
☐ How to write mathematical models for direct variation as an nth power	101, 102
☐ How to write mathematical models for inverse variation	103
☐ How to write mathematical models for joint variation	104
☐ How to use the *regression* feature of a graphing utility to find the equation of a least squares regression line	105

Review Exercises

3.1 In Exercises 1–4, find the quadratic function that has the indicated vertex and whose graph passes through the given point.

1. 

2. 

3. Vertex: $(1, -4)$; Point: $(2, -3)$
4. Vertex: $(2, 3)$; Point: $(-1, 6)$

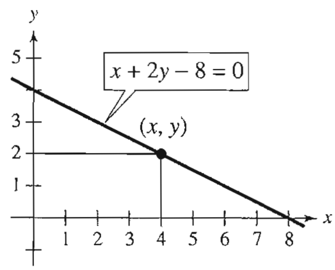
In Exercises 5 and 6, graph each function. Compare the graph of each function with the graph of $y = x^2$.

5. (a) $f(x) = 2x^2$
   (b) $g(x) = -2x^2$
   (c) $h(x) = x^2 + 2$
   (d) $k(x) = (x + 2)^2$
6. (a) $f(x) = x^2 - 4$
   (b) $g(x) = 4 - x^2$
   (c) $h(x) = (x - 3)^2$
   (d) $k(x) = \frac{1}{2}x^2 - 1$

In Exercises 7–18, write the quadratic function in standard form and sketch its graph. Identify the vertex and x-intercepts.

7. $g(x) = x^2 - 2x$
8. $f(x) = 6x - x^2$
9. $f(x) = x^2 + 8x + 10$
10. $h(x) = 3 + 4x - x^2$
11. $f(t) = -2t^2 + 4t + 1$
12. $f(x) = x^2 - 8x + 12$
13. $h(x) = 4x^2 + 4x + 13$
14. $f(x) = x^2 - 6x + 1$
15. $h(x) = x^2 + 5x - 4$
16. $f(x) = 4x^2 + 4x + 5$
17. $f(x) = \frac{1}{3}(x^2 + 5x - 4)$
18. $f(x) = \frac{1}{2}(6x^2 - 24x + 22)$

19. ***Numerical, Graphical, and Analytical Analysis*** A rectangle is inscribed in the region bounded by the x-axis, the y-axis, and the graph of $x + 2y - 8 = 0$, as shown in the figure.



   (a) Write the area A of the rectangle as a function of x.
   (b) Determine the domain of the function in the context of the problem.
   (c) Create a table showing possible values of x and the corresponding area of the rectangle.
   (d) Use a graphing utility to graph the area function. Use the graph to approximate the dimensions that will produce the maximum area.
   (e) Write the area function in standard form to find analytically the dimensions that will produce the maximum area.

20. ***Geometry*** The perimeter of a rectangle is 200 meters.
   (a) Draw a rectangle that gives a visual representation of the problem. Label the length and width in terms of x and y, respectively.
   (b) Write y as a function of x. Use the result to write the area as a function of x.
   (c) Of all possible rectangles with perimeters of 200 meters, find the dimensions of the one with the maximum area.

21. ***Maximum Revenue*** Find the number of units that produces a maximum revenue for

   $R = 800x - 0.01x^2$

   where R is the total revenue (in dollars) for a cosmetics company and x is the number of units produced.

22. ***Maximum Profit*** A real estate office handles an apartment building that has 50 units. When the rent is \$540 per month, all units are occupied. However, for each \$30 increase in rent, one unit becomes vacant. Each occupied unit requires an average of \$18 per month for service and repairs. What rent should be charged to obtain the maximum profit?

23. ***Minimum Cost*** A soft-drink manufacturer has daily production costs of

$$C = 70{,}000 - 120x + 0.055x^2$$

where C is the total cost (in dollars) and x is the number of units produced. How many units should be produced each day to yield a minimum cost?

24. ***Sociology*** The average age of the groom at a first marriage for a given age of the bride can be approximated by the model

$$y = -0.107x^2 + 5.68x - 48.5,$$
$$20 \le x \le 25$$

where y is the age of the groom and x is the age of the bride. For what age of the bride is the average age of the groom 26? (Source: U.S. Census Bureau)

3.2 **In Exercises 25–30, sketch the graphs of $y = x^n$ and the transformation.**

25. $y = x^3, \quad f(x) = -(x - 4)^3$

26. $y = x^3, \quad f(x) = -4x^3$

27. $y = x^4, \quad f(x) = 2 - x^4$

28. $y = x^4, \quad f(x) = 2(x - 2)^4$

29. $y = x^5, \quad f(x) = (x - 3)^5$

30. $y = x^5, \quad f(x) = \frac{1}{2}x^5 + 3$

In Exercises 31–34, determine the right-hand and left- hand behavior of the graph of the polynomial function.

31. $f(x) = -x^2 + 6x + 9$

32. $f(x) = \frac{1}{2}x^3 + 2x$

33. $g(x) = \frac{3}{4}(x^4 + 3x^2 + 2)$

34. $h(x) = -x^5 - 7x^2 + 10x$

In Exercises 35–40, find all the real zeros of the polynomial function. Determine the multiplicity of each zero.

35. $f(x) = 2x^2 + 11x - 21$

36. $f(x) = x(x + 3)^2$

37. $f(t) = t^3 - 3t$

38. $f(x) = x^3 - 8x^2$

39. $f(x) = -12x^3 + 20x^2$

40. $g(x) = x^4 - x^3 - 2x^2$

In Exercises 41–44, sketch the graph of the function by (a) applying the Leading Coefficient Test, (b) finding the zeros of the polynomial, (c) plotting sufficient solution points, and (d) drawing a continuous curve through the points.

41. $f(x) = -x^3 + x^2 - 2$

42. $g(x) = 2x^3 + 4x^2$

43. $f(x) = x(x^3 + x^2 - 5x + 3)$

44. $h(x) = 3x^2 - x^4$

In Exercises 45–48, use the Intermediate Value Theorem and the *table* feature of a graphing utility to find intervals one unit in length in which the polynomial function is guaranteed to have a zero. Adjust the table to approximate the zeros of the function. Use the *zero* or *root* feature of the graphing utility to verify your results.

45. $f(x) = 3x^3 - x^2 + 3$

46. $f(x) = 0.25x^3 - 3.65x + 6.12$

47. $f(x) = x^4 - 5x - 1$

48. $f(x) = 7x^4 + 3x^3 - 8x^2 + 2$

3.3 **In Exercises 49–54, use long division to divide.**

49. $\dfrac{24x^2 - x - 8}{3x - 2}$

50. $\dfrac{4x + 7}{3x - 2}$

51. $\dfrac{5x^3 - 13x^2 - x + 2}{x^2 - 3x + 1}$

52. $\dfrac{3x^4}{x^2 - 1}$

53. $\dfrac{x^4 - 3x^3 + 4x^2 - 6x + 3}{x^2 + 2}$

54. $\dfrac{6x^4 + 10x^3 + 13x^2 - 5x + 2}{2x^2 - 1}$

In Exercises 55–58, use synthetic division to divide.

55. $\dfrac{6x^4 - 4x^3 - 27x^2 + 18x}{x - 2}$

56. $\dfrac{0.1x^3 + 0.3x^2 - 0.5}{x - 5}$

57. $\dfrac{2x^3 - 19x^2 + 38x + 24}{x - 4}$

58. $\dfrac{3x^3 + 20x^2 + 29x - 12}{x + 3}$

In Exercises 59 and 60, use synthetic division to determine whether the given values of *x* are zeros of the function.

59. $f(x) = 20x^4 + 9x^3 - 14x^2 - 3x$

(a) $x = -1$ (b) $x = \frac{3}{4}$ (c) $x = 0$ (d) $x = 1$

60. $f(x) = 3x^3 - 8x^2 - 20x + 16$

(a) $x = 4$ (b) $x = -4$ (c) $x = \frac{2}{3}$ (d) $x = -1$

In Exercises 61 and 62, use synthetic division to find each specified value of the function.

61. $f(x) = x^4 + 10x^3 - 24x^2 + 20x + 44$

(a) $f(-3)$ (b) $f(-1)$

62. $g(t) = 2t^5 - 5t^4 - 8t + 20$

(a) $g(-4)$ (b) $g(\sqrt{2})$

In Exercises 63–66, (a) verify the given factor(s) of the function *f*, (b) find the remaining factors of *f*, (c) use your results to write the complete factorization of *f*, (d) list all real zeros of *f*, and (e) confirm your results by using a graphing utility to graph the function.

	Function	*Factor(s)*
63.	$f(x) = x^3 + 4x^2 - 25x - 28$	$(x - 4)$
64.	$f(x) = 2x^3 + 11x^2 - 21x - 90$	$(x + 6)$
65.	$f(x) = x^4 - 4x^3 - 7x^2 + 22x + 24$	$(x + 2)(x - 3)$
66.	$f(x) = x^4 - 11x^3 + 41x^2 - 61x + 30$	$(x - 2)(x - 5)$

3.4 **In Exercises 67–72, find all the zeros of the function.**

67. $f(x) = 3x(x - 2)^2$

68. $f(x) = (x - 4)(x + 9)^2$

69. $f(x) = x^2 - 9x + 8$

70. $f(x) = x^3 + 6x$

71. $f(x) = (x + 4)(x - 6)(x - 2i)(x + 2i)$

72. $f(x) = (x - 8)(x - 5)^2(x - 3 + i)(x - 3 - i)$

In Exercises 73 and 74, use the Rational Zero Test to list all possible rational zeros of *f*.

73. $f(x) = -4x^3 + 8x^2 - 3x + 15$

74. $f(x) = 3x^4 + 4x^3 - 5x^2 - 8$

In Exercises 75–80, find all the real zeros of the function.

75. $f(x) = x^3 - 2x^2 - 21x - 18$

76. $f(x) = 3x^3 - 20x^2 + 7x + 30$

77. $f(x) = x^3 - 10x^2 + 17x - 8$

78. $f(x) = x^3 + 9x^2 + 24x + 20$

79. $f(x) = x^4 + x^3 - 11x^2 + x - 12$

80. $f(x) = 25x^4 + 25x^3 - 154x^2 - 4x + 24$

In Exercises 81 and 82, find a polynomial with real coefficients that has the given zeros.

81. $\frac{2}{3}, 4, \sqrt{3}i$

82. $2, -3, 1 - 2i$

In Exercises 83–86, use the given zero to find all the zeros of the function.

	Function	*Zero*
83.	$f(x) = x^3 - 4x^2 + x - 4$	i
84.	$h(x) = -x^3 + 2x^2 - 16x + 32$	$-4i$
85.	$g(x) = 2x^4 - 3x^3 - 13x^2 + 37x - 15$	$2 + i$
86.	$f(x) = 4x^4 - 11x^3 + 14x^2 - 6x$	$1 - i$

In Exercises 87–90, find all the zeros of the function and write the polynomial as a product of linear factors.

87. $f(x) = x^3 + 4x^2 - 5x$

88. $g(x) = x^3 - 7x^2 + 36$

89. $g(x) = x^4 + 4x^3 - 3x^2 + 40x + 208$

90. $f(x) = x^4 + 8x^3 + 8x^2 - 72x - 153$

In Exercises 91–94, use a graphing utility to (a) graph the function, (b) determine the number of real zeros of the function, and (c) approximate the real zeros of the function to the nearest hundredth.

91. $f(x) = x^4 + 2x + 1$

92. $g(x) = x^3 - 3x^2 + 3x + 2$

93. $h(x) = x^3 - 6x^2 + 12x - 10$

94. $f(x) = x^5 + 2x^3 - 3x - 20$

In Exercises 95 and 96, use Descartes's Rule of Signs to determine the possible numbers of positive and negative zeros of the function.

95. $g(x) = 5x^3 + 3x^2 - 6x + 9$

96. $h(x) = -2x^5 + 4x^3 - 2x^2 + 5$

In Exercises 97 and 98, use synthetic division to verify the upper and lower bounds of the real zeros of *f*.

97. $f(x) = 4x^3 - 3x^2 + 4x - 3$

(a) Upper: $x = 1$

(b) Lower: $x = -\frac{1}{4}$

98. $f(x) = 2x^3 - 5x^2 - 14x + 8$

(a) Upper: $x = 8$

(b) Lower: $x = -4$

3.5 **99.** ***Data Analysis*** The federal minimum wage rates R (in dollars) in the United States for selected years from 1955 through 2000 are shown in the table. A linear model that approximates this data is

$$R = 0.099t - 0.08$$

where t represents the year, with $t = 5$ corresponding to 1955. (Source: U.S. Department of Labor)

Year	Wage rate, R
1955	0.75
1960	1.00
1965	1.25
1970	1.60
1975	2.10
1980	3.10
1985	3.35
1990	3.80
1995	4.25
2000	5.15

(a) Plot the actual data and the model on the same set of coordinate axes.

(b) How closely does the model represent the data?

100. ***Measurement*** You notice a billboard indicating that it is 2.5 miles or 4 kilometers to the next restaurant of a national fast-food chain. Use this information to find a linear model that relates miles to kilometers. Use the model to find the numbers of kilometers in 2 miles and 10 miles.

101. ***Energy*** The power P produced by a wind turbine is proportional to the cube of the wind speed S. A wind speed of 27 miles per hour produces a power output of 750 kilowatts. Find the output for a wind speed of 40 miles per hour.

102. ***Frictional Force*** The frictional force F between the tires and the road required to keep a car on a curved section of a highway is directly proportional to the square of the speed s of the car. If the speed of the car is doubled, the force will change by what factor?

In Exercises 103 and 104, find a mathematical model that represents the statement. (In each case, determine the constant of proportionality.)

103. y is inversely proportional to x. ($y = 9$ when $x = 5.5$.)

104. F is jointly proportional to x and the square root of y. ($F = 6$ when $x = 9$ and $y = 4$.)

105. ***Recording Media*** The table shows the numbers y (in millions) of CDs shipped in the United States in the years 1990 through 1999. (Source: Recording Industry Association of America)

Year	CDs shipped, y
1990	286.5
1991	333.3
1992	407.5
1993	495.4
1994	662.1
1995	722.9
1996	778.9
1997	753.1
1998	847.0
1999	938.9

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 1990.

(b) Use the *regression* feature of the graphing utility to find the equation of the least squares regression line that fits the data. Then graph the model and the scatter plot you found in part (a) in the same viewing window.

(c) Use the model to estimate the number of CDs that will be shipped in the year 2005.

(d) Interpret the meaning of the slope of the linear model in the context of the problem.

Synthesis

True or False? **In Exercises 106 and 107, determine whether the statement is true or false. Justify your answer.**

106. A fourth-degree polynomial can have -5, $-8i$, $4i$, and 5 as its zeros.

107. If y is directly proportional to x, then x is directly proportional to y.

Chapter Test

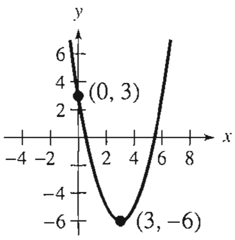


FIGURE FOR 3

The *Interactive* CD-ROM and *Internet* versions of this text offer Chapter Pre-Tests and Chapter Post-Tests, both of which have randomly generated exercises with diagnostic capabilities.

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

1. Describe how the graph of g differs from the graph of $f(x) = x^2$.
   (a) $g(x) = 2 - x^2$ (b) $g(x) = \left(x - \frac{3}{2}\right)^2$
2. Identify the vertex and intercepts of the graph of $y = x^2 + 4x + 3$.
3. Find an equation of the parabola shown in the figure at the left.
4. The path of a ball is given by $y = -\frac{1}{20}x^2 + 3x + 5$, where y is the height (in feet) of the ball and x is the horizontal distance (in feet) from where the ball was thrown.
   (a) Find the maximum height of the ball.
   (b) Which number determines the height at which the ball was thrown? Does changing this constant change the coordinates of the maximum height of the ball? Explain.
5. Determine the right-hand and left-hand behavior of the graph of the function $h(t) = -\frac{3}{4}t^5 + 2t^2$. Then sketch its graph.
6. Divide by long division.
   $$\frac{3x^3 + 4x - 1}{x^2 + 1}$$
7. Divide by synthetic division.
   $$\frac{2x^4 - 5x^2 - 3}{x - 2}$$
8. Use synthetic division to show that $x = \sqrt{3}$ is a zero of the function
   $f(x) = 4x^3 - x^2 - 12x + 3.$
   Use the result to factor the polynomial function completely and list all the real zeros of the function.

In Exercises 9 and 10, find all the real zeros of the function.

9. $g(t) = 2t^4 - 3t^3 + 16t - 24$
10. $h(x) = 3x^5 + 2x^4 - 3x - 2$

In Exercises 11 and 12, find a polynomial function with integer coefficients that has the given zeros.

11. $0, 3, 3 + i, 3 - i$
12. $1 + \sqrt{3}i, 1 - \sqrt{3}i, 2, 2$

In Exercises 13 and 14, find all the zeros of the function.

13. $f(x) = x^3 + 2x^2 + 5x + 10$
14. $f(x) = x^4 - 9x^2 - 22x - 24$

In Exercises 15–17, find a mathematical model that represents the statement. (In each case, determine the constant of proportionality.)

15. v varies directly as the square root of s. ($v = 24$ when $s = 16$.)
16. A varies jointly as x and y. ($A = 500$ when $x = 15$ and $y = 8$.)
17. b varies inversely as a. ($b = 32$ when $a = 1.5$.)

Proofs in Mathematics

These two pages contain proofs of four important theorems and polynomial functions. The first two theorems are from Section 3.3, and the second two theorems are from Section 3.4.

The Remainder Theorem *(p. 288)*

If a polynomial $f(x)$ is divided by $x - k$, the remainder is

$$r = f(k).$$

Proof

From the Division Algorithm, you have

$$f(x) = (x - k)q(x) + r(x)$$

and because either $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $x - k$, you know that $r(x)$ must be a constant. That is, $r(x) = r$. Now, by evaluating $f(x)$ at $x = k$, you have

$$\begin{aligned} f(k) &= (k - k)q(k) + r \\ &= (0)q(k) + r = r. \end{aligned}$$

To be successful in algebra, it is important that you understand the connection among *factors* of a polynomial, *zeros* of a polynomial function, and *solutions* or *roots* of a polynomial equation. The Factor Theorem is the basis for this connection.

The Factor Theorem *(p. 288)*

A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

Proof

Using the Division Algorithm with the factor $(x - k)$, you have

$$f(x) = (x - k)q(x) + r(x).$$

By the Remainder Theorem, $r(x) = r = f(k)$, and you have

$$f(x) = (x - k)q(x) + f(k)$$

where $q(x)$ is a polynomial of lesser degree than $f(x)$. If $f(k) = 0$, then

$$f(x) = (x - k)q(x)$$

and you see that $(x - k)$ is a factor of $f(x)$. Conversely, if $(x - k)$ is a factor of $f(x)$, division of $f(x)$ by $(x - k)$ yields a remainder of 0. So, by the Remainder Theorem, you have $f(k) = 0$.

The Fundamental Theorem of Algebra

The Linear Factorization Theorem is closely related to the Fundamental Theorem of Algebra. The Fundamental Theorem of Algebra has a long and interesting history. In the early work with polynomial equations, The Fundamental Theorem of Algebra was thought to have been not true, because imaginary solutions were not considered. In fact, in the very early work by mathematicians such as Abu al-Khwarizmi (c. 800 A.D.), negative solutions were also not considered.

Once imaginary numbers were accepted, several mathematicians attempted to give a general proof of the Fundamental Theorem of Algebra. These included Gottfried von Leibniz (1702), Jean D`Alembert (1740), Leonhard Euler (1749), Joseph-Louis Lagrange (1772), and Pierre Simon Laplace (1795). The mathematician usually credited with the first correct proof of the Fundamental Theorem of Algebra is Carl Friedrich Gauss, who published the proof in his doctoral thesis in 1799.

Linear Factorization Theorem (p. 293)

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers.

Proof

Using the Fundamental Theorem of Algebra, you know that f must have at least one zero, c_1. Consequently, $(x - c_1)$ is a factor of $f(x)$, and you have

$$f(x) = (x - c_1)f_1(x).$$

If the degree of $f_1(x)$ is greater than zero, you again apply the Fundamental Theorem to conclude that f_1 must have a zero c_2, which implies that

$$f(x) = (x - c_1)(x - c_2)f_2(x).$$

It is clear that the degree of $f_1(x)$ is $n - 1$, that the degree of $f_2(x)$ is $n - 2$, and that you can repeatedly apply the Fundamental Theorem n times until you obtain

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where a_n is the leading coefficient of the polynomial $f(x)$.

Factors of a Polynomial (p. 297)

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

Proof

To begin, you use the Linear Factorization Theorem to conclude that $f(x)$ can be *completely* factored in the form

$$f(x) = d(x - c_1)(x - c_2)(x - c_3) \cdot \cdot \cdot (x - c_n).$$

If each c_i is real, there is nothing more to prove. If any c_i is complex $(c_i = a + bi,$ $b \neq 0)$, then, because the coefficients of $f(x)$ are real, you know that the conjugate $c_j = a - bi$ is also a zero. By multiplying the corresponding factors, you obtain

$$\begin{aligned}(x - c_i)(x - c_j) &= [x - (a + bi)][x - (a - bi)]\\ &= x^2 - 2ax + (a^2 + b^2)\end{aligned}$$

where each coefficient is real.

P.S. Problem Solving

1. (a) Find the zeros of each quadratic function $g(x)$.
   (i) $g(x) = x^2 - 4x - 12$
   (ii) $g(x) = x^2 + 5x$
   (iii) $g(x) = x^2 + 3x - 10$
   (iv) $g(x) = x^2 - 4x + 4$
   (v) $g(x) = x^2 - 2x - 6$
   (vi) $g(x) = x^2 + 3x + 4$

   (b) For each function in part (a), use a graphing utility to graph $f(x) = (x - 2) \cdot g(x)$. Verify that $(2, 0)$ is an x-intercept of the graph of $f(x)$. Describe any similarities or differences in the behavior of the six functions at this x-intercept.

   (c) For each function in part (b), use the graph of $f(x)$ to approximate the other x-intercepts of the graph.

   (d) Describe the connections that you find among the results of parts (a), (b), and (c).

2. Quonset huts were developed during World War II. They were temporary housing structures that could be assembled quickly and easily. A Quonset hut is shaped like a half cylinder. A manufacturer has 600 square feet of material with which to build a Quonset hut.
   (a) The formula for the surface area of half a cylinder is $S = \pi r^2 + \pi r l$, where r is the radius and l is the length of the hut. Solve this equation for l when $S = 600$.
   (b) The formula for the volume of the hut is $V = \frac{1}{2}\pi r^2 l$. Write the volume V of the Quonset hut as a polynomial function of r.
   (c) Use the function you wrote in part (b) to find the maximum volume of a Quonset hut with a surface area of 600 square feet. What are the dimensions of the hut?

3. Show that if

   $$f(x) = ax^3 + bx^2 + cx + d$$

   then $f(k) = r$ where

   $$r = ak^3 + bk^2 + ck + d$$

   using long division. In other words, verify the Remainder Theorem for a third-degree polynomial function.

4. In 2000 B.C., the Babylonians solved polynomial equations by referring to tables of values. One such table gave the values of $y^3 + y^2$. To be able to use this table, the Babylonians sometimes had to manipulate the equation as shown below.

   $$ax^3 + bx^2 = c \qquad \text{Original equation}$$

   $$\frac{a^3x^3}{b^3} + \frac{a^2x^2}{b^2} = \frac{a^2c}{b^3} \qquad \text{Multiply each side by } \frac{a^2}{b^3}.$$

   $$\left(\frac{ax}{b}\right)^3 + \left(\frac{ax}{b}\right)^2 = \frac{a^2c}{b^3} \qquad \text{Rewrite.}$$

   Then they would find $(a^2c)/b^3$ in the $y^3 + y^2$ column of the table. Because they knew that the corresponding y-value was equal to $(ax)/b$, they could conclude that $x = (by)/a$.
   (a) Calculate $y^3 + y^2$ for $y = 1, 2, 3, \ldots, 10$. Record the values in a table.

   Use the table from part (a) and the method above to solve each equation.
   (b) $x^3 + x^2 = 252$
   (c) $x^3 + 2x^2 = 288$
   (d) $3x^3 + x^2 = 90$
   (e) $2x^3 + 5x^2 = 2500$
   (f) $7x^3 + 6x^2 = 1728$
   (g) $10x^3 + 3x^2 = 297$

   Using the methods from this chapter, verify your solution to each equation.

5. At a glassware factory, molten cobalt glass is poured into molds to make paperweights. Each mold is a rectangular prism whose height is 3 inches greater than the length of each side of the square base. A machine pours 20 cubic inches of liquid glass into each mold. What are the dimensions of the mold?

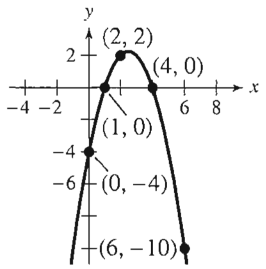
6. (a) Complete the table.

Function	Zeros	Sum of zeros	Product of zeros
$f_1(x) = x^2 - 5x + 6$			
$f_2(x) = x^3 - 7x + 6$			
$f_3(x) = x^4 + 2x^3 + x^2 + 8x - 12$			
$f_4(x) = x^5 - 3x^4 - 9x^3 + 25x^2 - 6x$			

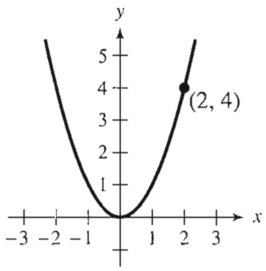
(b) Use the table to make a conjecture relating the sum of the zeros of a polynomial function with the coefficients of the polynomial function.

(c) Use the table to make a conjecture relating the product of the zeros of a polynomial function with the coefficients of the polynomial function.

7. The parabola shown in the figure has an equation of the form $y = ax^2 + bx + c$. Find the equation for this parabola by the following methods. (a) Find the equation analytically. (b) Use the *regression* feature of a graphing utility to find the equation.



8. One of the fundamental themes of calculus is to find the slope of the tangent line to a curve at a point. To see how this can be done, consider the point (2, 4) on the graph of the quadratic function $f(x) = x^2$.



(a) Find the slope of the line joining (2, 4) and (3, 9). Is the slope of the tangent line at (2, 4) greater than or less than the slope of the line through (2, 4) and (3, 9)?

(b) Find the slope of the line joining (2, 4) and (1, 1). Is the slope of the tangent line at (2, 4) greater than or less than the slope of the line through (2, 4) and (1, 1)?

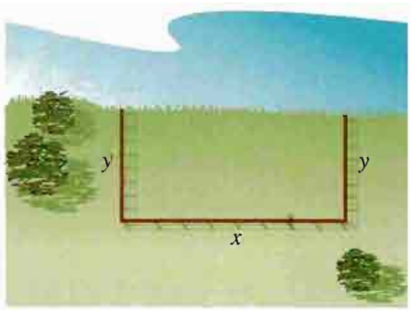
(c) Find the slope of the line joining (2, 4) and (2.1, 4.41). Is the slope of the tangent line at (2, 4) greater than or less than the slope of the line through (2, 4) and (2.1, 4.41)?

(d) Find the slope of the line joining (2, 4) and $(2 + h, f(2 + h))$ in terms of the nonzero number h.

(e) Evaluate the slope formula from part (d) for $h = -1$, 1, and 0.1. Compare these values with those in parts (a)–(c).

(f) What can you conclude the slope of the tangent line at (2, 4) to be? Explain your answer.

9. A rancher plans to fence a rectangular pasture adjacent to a river. The rancher has 100 meters of fence, and no fencing is needed along the river.

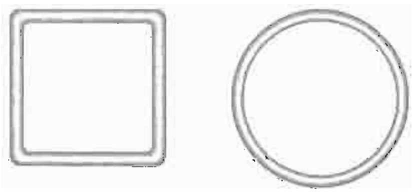


(a) Write the area as a function $A(x)$ of x, the length of the side of the pasture parallel to the river. What is the domain of $A(x)$?

(b) Graph the function $A(x)$ and estimate the dimensions that yield the maximum area of the pasture.

(c) Find the exact dimensions that yield the maximum area of the pasture by writing the quadratic function in standard form.

10. A wire 100 centimeters in length is cut into two pieces. One piece is bent to form a square and the other to form a circle. Let x equal the length of the wire used to form the square.



(a) Write the function that represents the combined area of the two figures.

(b) Determine the domain of the function.

(c) Find the value(s) of x that yield a maximum area and a minimum area.

(d) Explain your reasoning.

11. Find a formula for the polynomial division: $\dfrac{x^n - 1}{x - 1}$.