

P.1 – Review of Real Numbers and Their Properties

Objectives:

- Represent and classify real numbers
- Order real numbers and use inequalities
- Find the absolute values of real numbers and find the distance between two real numbers
- Evaluate algebraic expressions
- Use the basic rules and properties of algebra

REAL NUMBERS

Real numbers can describe quantities in everyday life such as age, miles per gallon, and population. Real numbers can be represented by symbols such as:

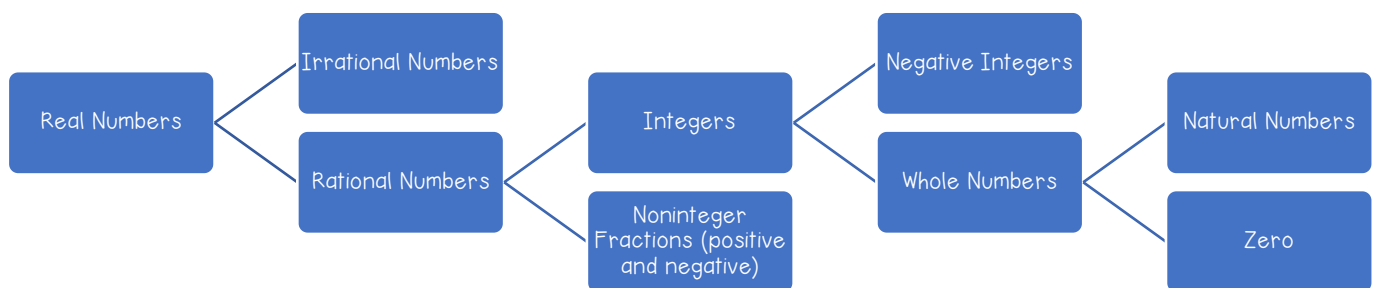
$$-5, 9, 0, \frac{4}{3}, 0.666, 28.21, \sqrt{2}, \pi, \text{ and } \sqrt[3]{-32}$$

A real number is **rational** if it can be written as the ratio $\frac{p}{q}$ of two integers, where $q \neq 0$. For example:

$$\frac{1}{3} = 0.3333 \dots = 0.\overline{3}, \frac{1}{8} = 0.125, \frac{125}{111} = 1.126126 \dots = 1.126\overline{126}$$

A real number that cannot be written as the ratio of two integers is called **irrational**. Irrational numbers have infinite nonrepeating decimal representations, like:

$$\sqrt{2} = 1.4142135 \dots \approx 1.41 \text{ and } \pi = 3.1415926 \dots \approx 3.14$$



I. Circle each numbers classification. A number can be classified as more than one type of number.

$-\sqrt{5}$:	Natural Number	Whole Number	Integer	Rational Number	Irrational Number
-1 :	Natural Number	Whole Number	Integer	Rational Number	Irrational Number
$-\frac{1}{3}$:	Natural Number	Whole Number	Integer	Rational Number	Irrational Number
0 :	Natural Number	Whole Number	Integer	Rational Number	Irrational Number
π :	Natural Number	Whole Number	Integer	Rational Number	Irrational Number
7 :	Natural Number	Whole Number	Integer	Rational Number	Irrational Number

ORDERING REAL NUMBERS

Definition of Order on the Real Number Line

If a and b are real numbers, a is less than b if $b - a$ is positive. The **order** of a and b is denoted by the inequality $a < b$. You can also describe the relationship by saying b is greater than a and writing the inequality $b > a$.

The inequality $a \leq b$ means that a is less than or equal to b , and the inequality $b \geq a$ means that b is greater than or equal to a .

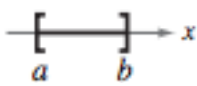
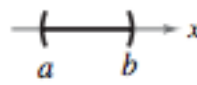
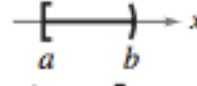
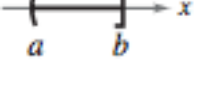
2. Describe the subset of real numbers represented by each inequality.

a. $x \leq 2$

b. $-2 \leq x < 3$


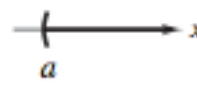
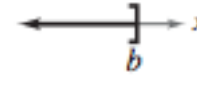
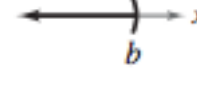

Inequalities can be used to describe subsets of real numbers called **intervals**. There are bounded intervals and unbounded intervals. In the bounded intervals, the real numbers a and b are the **endpoints** of each interval. The endpoints of a closed interval are included in the interval., whereas the endpoints of an open interval are not included in the interval.

Bounded Intervals on the Real Number Line

NOTATION	INTERVAL TYPE	INEQUALITY	GRAPH
$[a, b]$	Closed	$a \leq x \leq b$	
(a, b)	Open	$a < x < b$	
$[a, b)$		$a \leq x < b$	
$(a, b]$		$a < x \leq b$	

In unbounded intervals, the symbols ∞ , **positive infinity**, and $-\infty$, **negative infinity**, do not represent real numbers. They are symbols used to describe the unboundedness of some intervals.

Unbounded Intervals on the Real Number Line

NOTATION	INTERVAL TYPE	INEQUALITY	GRAPH
$[a, \infty)$		$x \geq a$	
(a, ∞)	Open	$x > a$	
$(-\infty, b]$		$x \leq b$	
$(-\infty, b)$	Open	$x < b$	
$(-\infty, \infty)$	Entire Real Line	$-\infty < x < \infty$	

3. Use inequality notation to describe each of the following:

a. c is at most 2

b. m is at least -3

c. All x in the interval $(-3,5]$

4. Give a verbal description of each interval.

a. $(-1,0)$

b. $[2, \infty)$

c. $(-\infty, 0)$

ABSOLUTE VALUE AND DISTANCE

The **absolute value** of a real number is its magnitude, or the distance between the origin and the point representing the real number on the real number line.

If a is a real number, then the absolute value of a is:

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a < 0 \end{cases}$$

NOTE: In this definition, the absolute value of a real number is NEVER negative.

5. Evaluate $\frac{|x|}{x}$ for

a. $x > 0$

b. $x < 0$

Properties of Absolute Values

1. $|a| \geq 0$

2. $|-a| = |a|$

3. $|ab| = |a||b|$

4. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, b \neq 0$

The **Law of Trichotomy** states that for any two real numbers, a and b , precisely one of the three relationships is possible:

$$a = b, a < b, \text{ or } a > b$$

6. Place the appropriate symbol ($<$, $>$, or $=$) between the pair of real numbers.

a. $|-4|$ $|3|$

b. $|-10|$ $|10|$

c. $-|-7|$ $|-7|$

Distance Between Two Points on the Real Line

Let a and b be real numbers. The distance between a and b is:

$$d(a, b) = |b - a| = |a - b|$$

7. Find the distance between -25 and 13 .

ALGEBRAIC EXPRESSIONS

One characteristic of algebra is the use of letters to represent numbers. The letters are called **variables**, and combinations of variables and numbers are called **algebraic expressions**. Some examples of algebraic expressions are:

$$5x \qquad 2x - 3 \qquad \frac{4}{x^2 + 2} \qquad 7x + y$$

An **algebraic expression** is a collection of letters (**variables**) and real numbers (**constants**) combined using the operations of addition, subtraction, and multiplication, division, and exponentiation.

The **terms** of an algebraic expression are the parts that are separated by addition or subtraction.

8. Circle the terms in the algebraic expressions.

a. $2x^2 - 6x + 9$

b. $4(x + 1)(x - 1) + (x - 2) - 3$

The **substitution principle** states, "if $a = b$, then b can replace a in any expression involving a ."

9. Find the value of the expression given the value of the variable.

a. $-3x + 5$ when $x = 3$

b. $3x^2 + 2x - 1$ when $x = -1$

BASIC RULES OF ALGEBRA

Basic Rules of Algebra

	PROPERTY	EXAMPLE
Commutative Property of Addition	$a + b = b + a$	$4x + x^2 = x^2 + 4x$
Commutative Property of Multiplication	$ab = ba$	$(4 - x)x^2 = x^2(4 - x)$
Associative Property of Addition	$(a + b) + c = a + (b + c)$	$(x + 5) + x^2 = x(5 + x^2)$
Associative Property of Multiplication	$(ab)c = a(bc)$	$(2x \cdot 3y)(8) = (2x)(3y \cdot 8)$
Distributive Property	$a(b + c) = ab + ac$	$3x(5 + 2x) = 3x \cdot 5 + 3x \cdot 2x$
Additive Identity Property	$a + 0 = a$	$5y^2 + 0 = 5y^2$
Multiplicative Identity Property	$a \cdot 1 = a$	$(4x^2)(1) = 4x^2$
Additive Inverse Property	$a + (-a) = 0$	$5x^3 + (-5x^3) = 0$
Multiplicative Inverse Property	$a \cdot \frac{1}{a} = 1, a \neq 0$	$(x^2 + 4) \left(\frac{1}{x^2 + 4} \right) = 1$

NOTE: Since subtraction is defined as "adding the opposite," the distributive property is also true for subtraction.

10. Identify the rule of algebra illustrated by the statement.

a. $(5x^3)2 = 2(5x^3)$

b. $\left(4x + \frac{1}{3}\right) - \left(4x + \frac{1}{3}\right) = 0$

c. $7x \cdot \frac{1}{7x} = 1, x \neq 0$

d. $(2 + 5x^2) + x^2 = 2 + (5x^2 + x^2)$

Properties of Negation and Equality

	PROPERTY	EXAMPLE
1.	$(-1)a = -a$	$(-1)7 = -7$
2.	$-(-a) = a$	$-(-6) = 6$
3.	$(-a)b = -(ab) = a(-b)$	$(-5)3 = -(5 \cdot 3) = 5(-3)$
4.	$(-a)(-b) = ab$	$(-1)(-x) = 2x$
5.	$-(a + b) = (-a) + (-b)$	$-(x + 8) = (-x) + (-8) = -x - 8$
6.	If $a = b$, then $a \pm c = b \pm c$.	$\frac{1}{2} + 3 = 0.5 + 3$
7.	If $a = b$, then $ac = bc$.	$4^2 \cdot 2 = 16 \cdot 2$
8.	If $a \pm c = b \pm c$, then $a = b$.	$1.4 - 1 = \frac{7}{5} - 1 \rightarrow 1.4 = \frac{7}{5}$
9.	If $ac = bc$ and $c \neq 0$, then $a = b$.	$3x = 3 \cdot 4 \rightarrow x = 4$

Properties of Zeros

Let a and b be real numbers, variables, or algebraic expressions.

- $a + 0 = a$ and $a - 0 = a$
- $a \cdot 0 = 0$
- $\frac{0}{a} = 0, a \neq 0$
- $\frac{a}{0}$ is undefined.
- Zero-Factor Property:** If $ab = 0$, then $a = 0$ or $b = 0$.

Properties and Operations of Fractions

Let $a, b, c,$ and d be real numbers, variables, or algebraic expressions such that $b \neq 0$ and $d \neq 0$.

- Equivalent Fractions:** $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$
- Rules of Signs:** $-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$ and $\frac{-a}{-b} = \frac{a}{b}$
- Generate Equivalent Fractions:** $\frac{a}{b} = \frac{ac}{bc}, c \neq 0$
- Add or Subtract with Like Denominators:** $\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$
- Add or Subtract with Unlike Denominators:** $\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$
- Multiply Fractions:** $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
- Divide Fractions:** $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}, c \neq 0$

II. Simplify the fractions.

a. $\frac{x}{5} = \frac{3}{3}$

b. $\frac{7}{x} \div \frac{3}{2}$

c. $\frac{x}{3} + \frac{2x}{5}$

d. $\frac{3}{5} \cdot \frac{x}{6}$

P.2 – Exponents and Radicals

Objectives:

- Use properties of exponents
- Use scientific notation to represent real numbers
- Use properties of radicals
- Simplify and combine radicals
- Rationalize denominators and numerators
- Use properties of rational exponents

INTEGER EXPONENTS

Properties of Exponents

Let a and b be real numbers, variables, or algebraic expressions, and let m and n be integers.

PROPERTY	EXAMPLE
1. $a^m a^n = a^{m+n}$	$3^2 \cdot 3^4 = 3^{2+4} = 3^6 = 729$
2. $\frac{a^m}{a^n} = a^{m-n}$	$\frac{x^7}{x^4} = x^{7-4} = x^3$
3. $a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n$	$y^{-4} = \frac{1}{y^4} = \left(\frac{1}{y}\right)^4$
4. $a^0 = 1, a \neq 0$	$(x^2 + 1)^0 = 1$
5. $(ab)^m = a^m b^m$	$(5x)^3 = 5^3 x^3 = 125x^3$
6. $(a^m)^n = a^{mn}$	$(y^3)^{-4} = y^{3(-4)} = y^{-12} = \frac{1}{y^{12}}$
7. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{2}{x}\right)^3 = \frac{2^3}{x^3} = \frac{8}{x^3}$
8. $ a^2 = a ^2 = a^2$	$ (-2)^2 = -2 ^2 = (2)^2 = 4$

1. Evaluate each exponential expression.

a. -3^4

b. $(-3)^4$

c. $3^2 \cdot 3$

d. $\frac{3^5}{3^8}$

2. Use the properties of exponents to simplify each expression.

a. $(2x^{-2}y^3)(-x^4y)$

b. $(4a^2b^3)^0$

c. $(-5z)^3(z^2)$

d. $\left(\frac{3x^4}{x^2y^2}\right)^2$

3. Rewrite each expression with positive exponents.

a. $2a^{-2}$

b. $\frac{3a^{-3}b^4}{15ab^{-1}}$

c. $\left(\frac{x}{10}\right)^{-1}$

d. $(-2x^2)^3(4x^3)^{-1}$

SCIENTIFIC NOTATION

Exponents provide an efficient way of writing and computing with very large (or very small) numbers. It is convenient to write such numbers in **scientific notations**. This notation is in the form $\pm c \times 10^n$, where $1 \leq c < 10$ and n is an integer.

The number of gallons of water on Earth can be written in scientific notation as:

$$3.59 \times 100,000,000,000,000,000,000 = 3.59 \times 10^{20}$$

The positive exponent 20 indicates that the number is large (10 or more) and that the decimal point has been moved 20 places.

The mass (in grams) of one electron is approximately:

$$9.0 \times 10^{-28} = 0.0000000000000000000000000009$$

The negative exponent -28 indicates that the number is small (less than 1) and that the decimal point has been moved 28 places.

4. Write each number in scientific notation.

a. 0.0000782

b. 45,850

5. Write each number in decimal notation.

a. -2.718×10^{-3}

b. 1.345×10^2

6. Evaluate $(24,000,000,000)(0.00000012)(300,000)$

RADICALS AND THEIR PROPERTIES

A **square root** of a number is one of its two equal factors. In a similar way, a **cube root** of a number is one of its three equal factors.

Definition of nth Root of a Number

Let a and b be real numbers and let $n \geq 2$ be a positive integer.

If $a = b^n$, then b is an **nth root of a** .

If $n = 2$, the root is a square root. If $n = 3$, the root is a cube root.

Principal nth Root of a Number

Let a be a real number that has at least one nth root.

The **principal nth root of a** is the nth root that has the same sign as a . It is denoted by a **radical symbol** $\sqrt[n]{a}$.

The positive integer n is the **index** of the radical, and the number a is the **radicand**. If $n = 2$, omit the index and write \sqrt{a} rather than $\sqrt[2]{a}$.

7. Evaluate each expression involving radicals.

a. $\sqrt{-144}$

b. $-\sqrt{144}$

c. $-\sqrt[3]{\frac{8}{27}}$

d. $\sqrt{\frac{25}{64}}$

Here are some generalizations about the n th roots of real numbers!

GENERALIZATIONS ABOUT n TH ROOTS OF REAL NUMBERS			
Real Number a	Integer n	Root(s) of a	Example
$a > 0$	$n > 0, n$ is even	$\sqrt[n]{a}, -\sqrt[n]{a}$	$\sqrt[4]{81} = 3, -\sqrt[4]{81} = -3$
$a > 0$ or $a < 0$	n is odd	$\sqrt[n]{a}$	$\sqrt[3]{-8} = -2$
$a < 0$	n is even	No real roots	$\sqrt{-4}$ is not a real number
$a = 0$	n is even or odd	$\sqrt[n]{0} = 0$	$\sqrt[5]{0} = 0$

Integers such as 1, 4, 9, 16, 25, and 36 are called **perfect squares** because they have integer square roots. Similarly, integers such as 1, 8, 27, 64, and 125 are called **perfect cubes** because they have integer cube roots.

Properties of Radicals

Let a and b be real numbers, variables, or algebraic expressions such that the indicated roots are real numbers, and let m and n be positive integers.

PROPERTY

EXAMPLE

$$1. \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

$$\sqrt[3]{8^2} = (\sqrt[3]{8})^2 = (2)^2 = 4$$

$$2. \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

$$\sqrt{5} \cdot \sqrt{7} = \sqrt{5 \cdot 7} = \sqrt{35}$$

$$3. \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}, b \neq 0$$

$$\frac{\sqrt[4]{27}}{\sqrt[4]{9}} = \sqrt[4]{\frac{27}{9}} = \sqrt[4]{3}$$

$$4. \sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

$$\sqrt[3]{\sqrt{10}} = \sqrt[6]{10}$$

$$5. (\sqrt[n]{a})^n = a$$

$$(\sqrt{3})^2 = 3$$

$$6. \text{For } n \text{ even, } \sqrt[n]{a^n} = |a|$$

$$\sqrt{(-12)^3} = |-12| = 12$$

$$\text{For } n \text{ odd, } \sqrt[n]{a^n} = a$$

$$\sqrt[3]{(-12)^3} = -12$$

8. Use the properties of radicals to simplify each expression.

a. $\sqrt[3]{x^2} \cdot \sqrt[3]{x}$

b. $\sqrt[3]{125^2}$

c. $\frac{\sqrt{125}}{\sqrt{5}}$

d. $\sqrt{\sqrt{x}}$

SIMPLIFYING RADICALS

An expression involving radicals is in **simplest form** when the following conditions are satisfied.

1. All possible factors have been removed from the radical.
2. All fractions have radical-free denominators (accomplished by a process called rationalizing the denominator).
3. The index of the radical is reduced.

To simplify a radical, factor the radicand into factors whose exponents are multiples of the index. The roots of these factors are written outside the radical, and the "leftover" factors make up the new radicand.

9. Simplify the radicals.

a. $\sqrt{32}$

b. $\sqrt[3]{250}$

c. $\sqrt{24a^5}$

d. $\sqrt[3]{-135x^3}$

Radical expressions can be combined (added or subtracted) if they are **like radicals**, meaning they have the same index and radicand. To determine if radicals can be combined, simplify each radical first.

10. Combine the radicals.

a. $3\sqrt{8} + \sqrt{18}$

b. $\sqrt[3]{81x^5} - \sqrt[3]{24x^2}$

RATIONALIZING DENOMINATORS AND NUMERATORS

To rationalize a denominator or numerator of the form $a - b\sqrt{m}$ or $a + b\sqrt{m}$, multiply both numerator and denominator by a **conjugate**: $a - b\sqrt{m}$ and $a + b\sqrt{m}$ are conjugates of each other. If $a = 0$, then the rationalizing factor for \sqrt{m} is itself. For cube roots, choose a rationalizing factor that generates a perfect cube.

11. Rationalize the denominator.

a. $\frac{5}{3\sqrt{2}}$

b. $\frac{1}{\sqrt[3]{25}}$

c. $\frac{2}{3+\sqrt{7}}$

d. $\frac{8}{\sqrt{6}-\sqrt{2}}$

Sometimes it is necessary to rationalize the numerator of an expression.

12. Rationalize the numerator.

a. $\frac{\sqrt{5}-\sqrt{7}}{2}$

b. $\frac{2-\sqrt{2}}{3}$

c. $\frac{2-\sqrt{11}}{6}$

d. $\frac{\sqrt[3]{2x^2}}{\sqrt[3]{5y}}$

RATIONAL EXPONENTS

Definition of Rational Exponents

If a is a real number and n is a positive integer such that the principal n th root of a exists, then $a^{\frac{1}{n}}$ is defined as

$$a^{\frac{1}{n}} = \sqrt[n]{a}, \text{ where } \frac{1}{n} \text{ is the rational exponent of } a.$$

Moreover, if m is a positive integer that has no common factor with n , then

$$a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = (\sqrt[n]{a})^m \text{ and } a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = \sqrt[n]{a^m}.$$

The numerator of a rational exponent denotes the power to which the base is raised, and the denominator denotes the index or the root to be taken.

NOTE: When working with rational exponents, the properties of integer exponents still apply.

13. Change the expression from radical form to exponential form.

a. $\sqrt[3]{27}$

b. $\sqrt{x^3y^5z}$

c. $3x\sqrt[3]{x^3}$

d. $\frac{1}{\sqrt[3]{6x^2}}$

14. Change the expression from exponential form to radical form.

a. $(x^2 + 7)^{-1/2}$

b. $-3b^{1/3}c^{2/3}$

c. $a^{0.75}$

d. $(x^2)^{2/5}$

15. Simplify the expression.

a. $(-125)^{-2/3}$

b. $(5x^2y^{3/2})(-3x^{-1/3}y^{-3/5})$

c. $\sqrt[3]{\sqrt[4]{27}}$

d. $(3x + 2)^{5/2}(3x + 2)^{-1/2}$

P.3 – Polynomials and Special Products

Objectives:

- Write polynomials in standard form
- Add, subtract, and multiply polynomials
- Use special products to multiply polynomials
- Use polynomials to solve real-life problems

The most common type of algebraic expression is the **polynomial**. Some examples are:

$$2x + 5, \quad 3x^4 - 7x^2 + 2x + 4, \quad \text{and} \quad 5x^2y^2 - xy + 3$$

Definition of a Polynomial in x

Let $a_0, a_1, a_2, \dots, a_n$ be real numbers and let n be a nonnegative integer. A polynomial in x is an expression of the form:

$$a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$$

where $a_n \neq 0$. The polynomial is of **degree n** , a_n is the **leading coefficient**, and a_0 is the **constant term**.

Polynomials with one, two, and three terms are called **monomials**, **binomials**, and **trinomials**, respectively. In **standard form**, a polynomial is written with descending powers of x .

1. Write each polynomial in standard form and state what the degree and leading coefficient is.

POLYNOMIAL	STANDARD FORM	DEGREE	LEADING COEFFICIENT
a. $4x^2 - 5x^7 - 2 + 3x$			
b. $4 - 9x^2$			
c. 8			

OPERATIONS WITH POLYNOMIALS

You can add and subtract polynomials in much the same way you add and subtract real numbers. Simply add or subtract the **like terms** (terms having the same variables to the same powers) by adding their coefficients.

2. Add or subtract the polynomials.

a. $(5x^3 - 7x^2 - 3) + (x^3 + 2x^2 - x + 8)$

b. $(2x^3 - x + 3) - (x^2 - 2x - 3)$

3. Multiply each polynomial.

a. $(3x - 1)(x - 5)$

c. $(x^2 - 2x + 2)(x^2 + 2x + 2)$

SPECIAL PRODUCTS

Some binomial products have special forms that occur frequently.

Special Products

Let u and v be real numbers, variables, or algebraic expressions.

SPECIAL PRODUCT

EXAMPLE

Sum and Difference of Same Terms

$$(u + v)(u - v) = u^2 - v^2$$

$$\begin{aligned}(x + 4)(x - 4) &= x^2 - 4^2 \\ &= x^2 - 16\end{aligned}$$

Square of a Binomial

$$(u + v)^2 = u^2 + 2uv + v^2$$

$$\begin{aligned}(x + 3)^2 &= x^2 + 2(x)(3) + 3^2 \\ &= x^2 + 6x + 9\end{aligned}$$

$$(u - v)^2 = u^2 - 2uv + v^2$$

$$\begin{aligned}(3x - 2)^2 &= (3x)^2 - 2(3x)(2) + 2^2 \\ &= 9x^2 - 12x + 4\end{aligned}$$

Cube of a Binomial

$$(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$$

$$\begin{aligned}(x + 2)^3 &= x^3 + 3x^2(2) + 3x(2^2) + 2^3 \\ &= x^3 + 6x^2 + 12x + 8\end{aligned}$$

$$(u - v)^3 = u^3 - 3u^2v + 3uv^2 - v^3$$

$$\begin{aligned}(x - 1)^3 &= x^3 - 3x^2(1) + 3x(1^2) - 1^3 \\ &= x^3 - 3x^2 + 3x - 1\end{aligned}$$

4. Find the product of the two polynomials.

a. $(3x - 2)(3x + 2)$

b. $(x + 10)^2$

c. $(4x - 1)^3$

d. $(x - 2 + 3y)(x - 2 - 3y)$

APPLICATION

5. If a square has a side length of $x + 2$ inches, find the perimeter.

6. Find the area of the same square in #5.

7. a) An open box is made by cutting squares from the corners of a piece of metal that is 12 inches by 10 inches. The edge of each cut-out square is x inches. Find the volume of the box.

b) When $x = 2$ and $x = 3$, which value of x gives the largest volume?

P.4 – Factoring Polynomials

Objectives:

- Remove common factors from polynomials
- Factor special polynomial forms
- Factor trinomials as a product of two binomials
- Factor polynomials by grouping

POLYNOMIALS WITH COMMON FACTORS

The process of writing a polynomial as a product of is called **factoring**. A polynomial is completely factored when each of its factors is **prime**.

The simplest form of factoring involves a polynomial that can be the product of a monomial and another polynomial. The technique used for this is the distributive property, in the reverse direction.

$$ab + ac = a(b + c)$$

NOTE: Removing (factoring out) the greatest common factor (GCF) is the first step in completely factoring a polynomial.

1. Factor out the GCF of each expression.

a. $5x^3 - 15x^2$

b. $-3 + 6x - 4x^3$

c. $18x^2y - 42xy^5$

d. $(x + 1)(x^2) - (x + 1)(2)$

FACTORIZING SPECIAL POLYNOMIAL FORMS

After factoring out the GCF, some polynomials can still be factored. Some of the polynomials will have special forms, like the ones you learned in lesson P.3.

One of the easiest special polynomial forms is the difference of two squares. To recognize a difference of two squares, look for a polynomial with two terms, a subtraction sign, exponents with even powers, and coefficients that are perfect squares.

$$(u + v)(u - v) = u^2 - v^2$$

2. Factor completely.

a. $18x^2 - 8$

b. $4x^4 + 25$

c. $16x^4 - 81$

d. $x^4 - 36$

The next two formulas show the sum and difference of cubes.

SUM: $u^3 + v^3 = (u + v)(u^2 - uv + v^2)$

DIFFERENCE: $u^3 - v^3 = (u - v)(u^2 + uv + v^2)$

NOTE: To remember the order of the signs, think SOAP (same sign, opposite sign, and always positive)

3. Factor completely.

a. $64x^3 - 1$

b. $5y^3 + 135$

c. $x^3 + 216$

d. $54x^3 - 2$

FACTORING BY GROUPING

Sometimes polynomials with can be factored by a method called **factoring by grouping**. It is not always obvious which terms to group, and sometimes several different groupings will work.

Step 1: Group the first two terms together, the group the last two terms together.

Step 2: Factor out the GCF from each separate group.

- If there is no GCF, factor out a 1.

- NOTE: The remaining parentheses should be identical. If they are not, try rearranging the original polynomial.

Step 3: Factor out the common binomial.

4. Factor by grouping.

a. $x^3 - 2x^2 - 3x + 6$

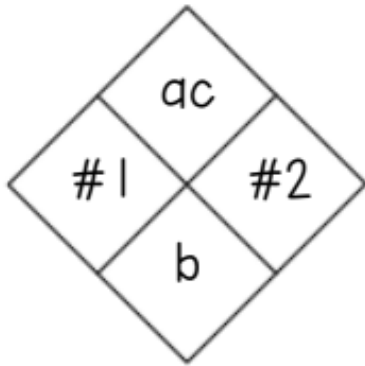
b. $6x^3 - 9x^2 + 4x - 6$

c. $x^3 - 3x^2 + 3x - 1$

d. $2x^3 - 3x^2 - 8x + 12$

TRINOMIALS WITH BINOMIAL FACTORS

When factoring trinomials, you can use the factoring by grouping method. To do so, you follow these steps.



Step 1: The TOP number is the product of "ac," the BOTTOM number is "b."

- Make sure the trinomial is in descending order!

Step 2: Find two numbers when you multiply them, you get "ac," and when you add them, you get "b." These two numbers are #1 and #2.

Step 3: Use #1 and #2, as coefficients to split the middle term of the original problem. The first and last term always stay the same!

Step 4: Factoring by GROUPING.

5. Factor by grouping.

a. $x^2 - 7x + 12$

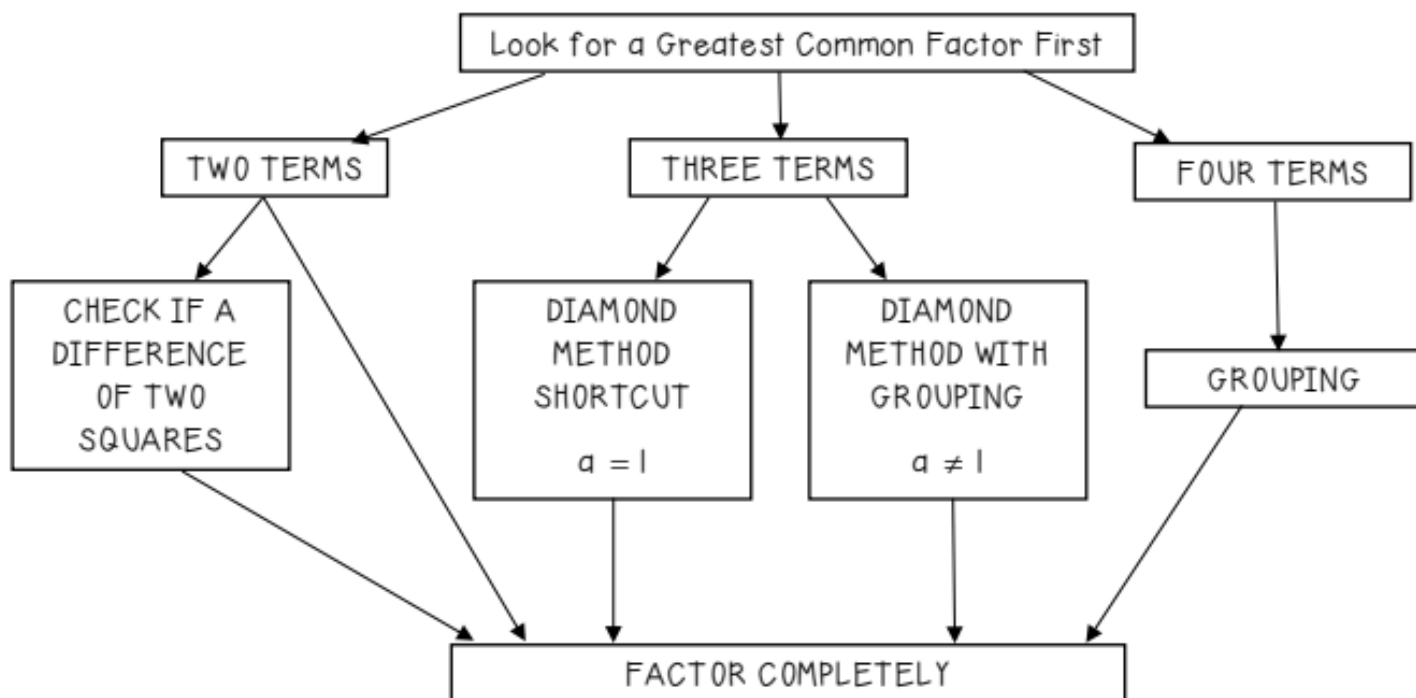
b. $2x^2 + x - 15$

c. $2x^2 + 5x - 3$

d. $1 - 4x + 3x^2$

e. $5x^2 - 11x - 12$

f. $x^2 + 2x + 1$



6. Factor completely.

a. $4x^2 - 20x + 16$

b. $x^3 - 2x^2 - 4x + 8$

c. $162x^2 - 2$

d. $9x^3 - 39x^2 - 30x$

e. $4x^3 + 4x^2 - 9x - 9$

f. $2x^2 + 32$

ZERO-PRODUCT PROPERTY

The zero-product property is how we solve for our variable when solving quadratic equation functions. If the product of two quantities equal zero, at least one of the quantities equals zero.

7. Use the zero-product property to solve the following equations.

a. $(x - 3)(x + 7) = 0$

b. $x(x - 5) = 0$

c. $(3x - 10)^2 = 0$

d. $5(3x + 2)(4x - 1) = 0$

SOLVING QUADRATIC FUNCTIONS BY FACTORING

To solve a quadratic equation, you have to use the zero-product property. However, it can only be used if the equation is equal to zero!

Step 1: Set the quadratic equation equal to zero

Step 2: Factor the left-hand side COMPLETELY

Step 3: Set each factor equal to zero

Step 4: Solve for the variable

8. Solve each quadratic equation

a. $x^2 + 10x + 9 = 0$

b. $t^2 - 2t = 15$

c. $2d^2 - 16d + 37 = 5$

d. $18x^2 = 24x$

P.5 – Rational Expressions

Objectives:

- Find domains of algebraic expressions
- Simplify rational expressions
- Add, subtract, multiply, and divide rational expressions
- Simplify complex fractions and rewrite difference quotients

DOMAIN OF AN ALGEBRAIC EXPRESSION

The set of real numbers for which an algebraic expression is defined is the **domain** of the expression.

Two algebraic expressions are **equivalent** if they have the same domain and yield the same values for all numbers in their domain.

I. Find the domain of each algebraic expression.

a. $4x^3 + 3$

b. $\sqrt{x + 7}$

c. $\frac{1-x}{x}$

SIMPLIFYING RATIONAL EXPRESSIONS

The quotient of two algebraic expressions is a fractional expression. Moreover, the quotient of two polynomials is a **rational expression**.

A fraction is in its simplest form if its numerator and denominator have no factors in common (aside from ± 1). To write a fraction in simplest form, divide out the common factors.

$$\frac{a \cdot c}{b \cdot c} = \frac{a}{b}, \quad c \neq 0$$

NOTE: To be successful in simplifying rational expressions relies on your ability to factor polynomials. Factor the numerator AND denominator completely before concluding that the numerator and denominator have no factors in common.

2. Simplify each rational expression.

a. $\frac{4x+12}{x^2-3x-18}$

b. $\frac{3x^2-x-2}{4-4x-x^2}$

Rational expressions can also have **excluded values**. Excluded values are x-values that cannot be included in the domain because it makes the expression undefined. A rational expression is undefined when the denominator is equal to zero. Therefore, excluded values are based off the denominator.

3. What are the excluded values of the rational expressions in #2?

a. $\frac{x^2+4x-12}{3x-6}$

b. $\frac{12+x-x^2}{2x^2-9x+4}$

OPERATIONS WITH RATIONAL EXPRESSIONS

To multiply or divide rational expressions, use the properties of fractions discussed in lesson P. I. Remember, to divide fractions, you invert the divisor and multiply.

4. Multiply each rational expression.

a. $\frac{15x^2+5x}{x^3-3x^2-18x} \cdot \frac{x^2-2x-15}{3x^2-8x-3}$

b. $\frac{2x^2+7x-15}{4x^2-8x+3} \cdot \frac{2x^2+x-1}{x^2+6x+5}$

5. Divide each rational expression.

a. $\frac{x^3-1}{x^2-1} \div \frac{x^2+x+1}{x^2+2x+1}$

b. $\frac{12x^2-22x+8}{3x} \div \frac{3x^2+2x-8}{2x^2+4x}$

To add or subtract rational expressions, you can use the least common denominator (LCD) method or the basic definition $\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$, $b \neq 0, d \neq 0$. This definition provides an efficient way of adding or subtracting TWO fractions that have no common factors in their denominators.

6. Combine the rational expressions.

a. $\frac{x}{2x-1} - \frac{1}{x+2}$

b. $\frac{4}{x} - \frac{x+5}{x^2-4} + \frac{4}{x+2}$

c. $\frac{x+1}{x^2-2x-35} + \frac{x+6}{x^2+7x+10}$

d. $\frac{2x}{1-2x} + \frac{3x}{2x+1} - \frac{3}{4x^2-1}$

e. $\frac{x-23}{x^2-x-20} - \frac{2}{5-x}$

f. $\frac{x-2}{(a+2)^2} + \frac{x}{5x+10} - \frac{1}{25}$

COMPLEX FRACTIONS AND THE DIFFERENCE QUOTIENT

Fractional expressions with separate fractions in the numerator, denominator, or both are called **complex fractions**.

To simplify a complex fraction, combine the fractions in the numerator into a single fraction and then combine the fractions in the denominator into a one fraction. Then invert the denominator and multiply.

7. Simplify the complex fraction.

a. $\frac{\frac{1}{x+2}+1}{\frac{x}{3}-1}$

b. $\frac{2+\frac{10}{x}}{1-\frac{25}{x^2}}$

To simplify an expression with negative exponents, factor out the common factor with the smaller exponent.

NOTE: When you factor, you subtract exponents.

8. Simplify the expression and make sure all the exponents are positive.

a. $(x-1)^{-\frac{1}{3}} - x(x-1)^{-\frac{4}{3}}$

b. $\frac{(4-x^2)^{\frac{1}{2}} + x^2(4-x^2)^{-\frac{1}{2}}}{4-x^2}$

9. Rewrite the expression by rationalizing the numerator.

a. $\frac{\sqrt{9+h}-3}{h}$

b. $\frac{\sqrt{x+2}-\sqrt{x}}{2}$

P.6 – Errors and the Algebra of Calculus

Objectives:

- Avoid common algebraic errors
- Recognize and use algebraic techniques that are common in calculus

ALGEBRAIC ERRORS TO AVOID

There are five lists of common algebraic errors: errors involving parentheses, errors involving fractions, errors involving exponents, errors involving radicals, and errors involving dividing out.

Errors Involving Parentheses

POTENTIAL ERROR	CORRECT FORM	COMMENT
$a - (x - b) \neq a - x - b$		
$(a + b)^2 \neq a^2 + b^2$		
$\left(\frac{1}{2}a\right)\left(\frac{1}{2}b\right) \neq \frac{1}{2}(ab)$		
$(3x + 6)^2 \neq 3(x + 2)^2$		

Errors Involving Fractions

POTENTIAL ERROR	CORRECT FORM	COMMENT
$\frac{a}{x+b} \neq \frac{a}{x} + \frac{a}{b}$		
$\frac{\left(\frac{x}{a}\right)}{b} \neq \frac{bx}{a}$		
$\frac{1}{a} + \frac{1}{b} \neq \frac{1}{a+b}$		
$\frac{1}{3x} \neq \frac{1}{3}x$		
$\left(\frac{1}{3}\right)x \neq \frac{1}{3x}$		
$\left(\frac{1}{x}\right) + 2 \neq \frac{1}{x+2}$		

Errors Involving Exponents

POTENTIAL ERROR	CORRECT FORM	COMMENT
$(x^2)^3 \neq x^5$		
$x^2 \cdot x^3 \neq x^6$		
$2x^3 \neq (2x)^3$		
$\frac{1}{x^2 - x^3} \neq x^{-2} - x^{-3}$		

Errors Involving Radicals

POTENTIAL ERROR	CORRECT FORM	COMMENT
$\sqrt{5x} \neq 5\sqrt{x}$		
$\sqrt{x^2 + a^2} \neq x + a$		
$\sqrt{-x + a} \neq -\sqrt{x - a}$		

Errors Involving Dividing Out

POTENTIAL ERROR	CORRECT FORM	COMMENT
$\frac{a + bx}{a} \neq 1 + bx$		
$\frac{a + ax}{a} \neq a + x$		
$1 + \frac{x}{2x} \neq 1 + \frac{1}{x}$		

1. Describe and correct the error.

a. $2x - (3y + 4) = 2x - 3y + 4$

b. $(5z)(6z) = 30z$

c. $a\left(\frac{x}{y}\right) = \frac{ax}{ay}$

d. $(4x)^2$

e. $\sqrt{x+9} = \sqrt{x} + 3$

f. $\frac{6x+y}{6x-y} = \frac{x+y}{x-y}$

g. $\frac{1}{a^{-1}+b^{-1}} = \left(\frac{1}{a+b}\right)^{-1}$

h. $\frac{1}{2y} = \left(\frac{1}{2}\right)y$

P.7 – Graphical Representation of Data

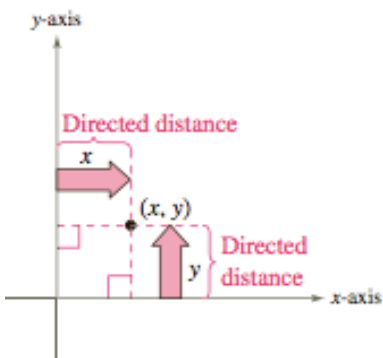
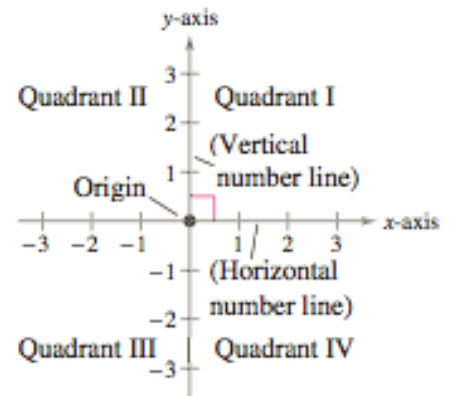
Objectives:

- Plot points in the Cartesian plane
- Use the Distance Formula to find the distance between two points
- Use the Midpoint Formula to find the midpoint of a line segment
- Use a coordinate plane to model and solve real-life problems

THE CARTESIAN PLANE

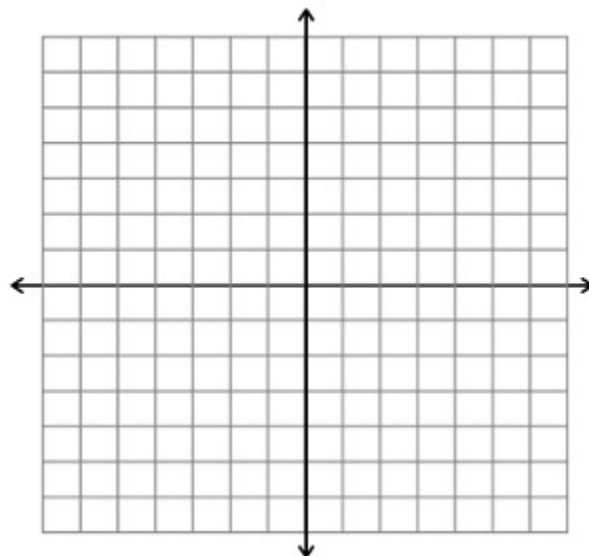
Just as you represent real numbers by points on a number line, you can represent ordered pairs of real numbers by points in a plane called the **rectangular coordinate system**, or the **Cartesian plane**.

The Cartesian plane is formed by using two real number lines intersecting at right angles. The horizontal real number line is usually called the **x-axis**, and the vertical real number line is usually called the **y-axis**. The point of intersection of these two axes is the **origin**, and the two axes divide the plane into four parts called **quadrants**.



Each point in the plane corresponds to an **ordered pair** (x, y) of real numbers x and y , called **coordinates** of the point. The **x-coordinate** represents the directed distance from the **y-axis** to the point, and the **y-coordinate** represents the directed distance from the **x-axis** to the point.

I. Plot the points $(-3, 2)$, $(4, -2)$, $(3, 1)$, $(0, -2)$, and $(-1, -2)$ on the Cartesian plane.

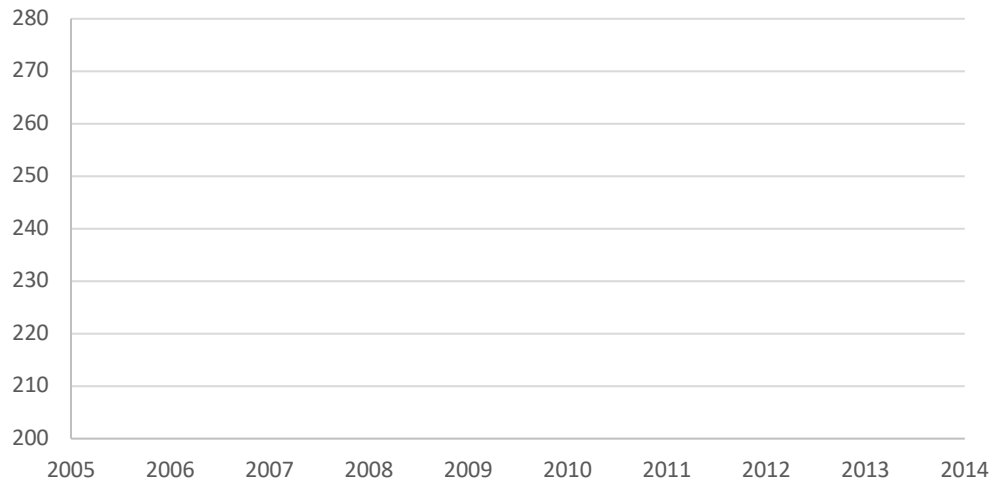


With rectangular coordinate systems, it allows you to see relationships between two variables.

2. The table below shows the numbers N (in millions) of subscribers to a cellular telecommunication service in the United States from 2005 through 2014, where t represents the year. Sketch a scatter plot of the data.

Subscribers to a Cellular
Telecommunications Service

Year, t	Subscribers, N
2005	233.1
2006	253.8
2007	266.8
2008	268.5
2009	249.2
2010	250.4
2011	238.1
2012	230.1
2013	230.4
2014	232.2



THE DISTANCE FORMULA

Suppose you want to determine the distance d between two points (x_1, y_1) and (x_2, y_2) in the plane. Within these two points, a right triangle can be formed. The length of the vertical side of the triangle is $|y_2 - y_1|$, and the length of the horizontal side is $|x_2 - x_1|$.

The Distance Formula

The distance d between the points (x_1, y_1) and (x_2, y_2) in the plane is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

3. Find the distance between the given points.

a. $(3,1)$ and $(-3,0)$

b. $(3,8)$ and $(9,10)$

4. Show that the given points are vertices of a right triangle.

a. $(2, -1)$, $(5, 5)$, and $(6, -3)$

b. $(1, 2)$, $(5, 4)$, and $(-3, 0)$

THE MIDPOINT FORMULA

To find the **midpoint** of the line segment that joins two points in a coordinate plane, you can find the average values of the respective coordinates of the two endpoints using the **midpoint formula**.

The Midpoint Formula

The midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) is given by the formula:

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

5. Find the midpoint of the line segment joining the given points.

a. $(-2, 8)$, $(4, -10)$

b. $(2, -1)$, $(-6, 0)$

APPLICATION

6. A football quarterback throws a pass from the 10-yard line, 10 yards from the sideline. The pass caught by the wide receiver on the 32-yard line, 25 yards from the same sideline. How long is the pass?

7. Starbucks Corporation had annual sales of approximately \$13.3 billion in 2012 and \$16.4 billion in 2014. Without knowing any additional information, what would you estimate the 2013 sales to have been?